

Mining billion-scale tensors: algorithms and discoveries

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Abstract How can we analyze large-scale real-world data with various attributes? Many real-world data (e.g., network traffic logs, web data, social networks, knowledge bases, and sensor streams) with multiple attributes are represented as multi-dimensional arrays, called tensors. For analyzing a tensor, tensor decompositions are widely used in many data mining applications: detecting malicious attackers in network traffic logs (with source IP, destination IP, port-number, timestamp), finding telemarketers in a phone call history (with sender, receiver, date), and identifying interesting concepts in a knowledge base (with subject, object, relation). However, current tensor decomposition methods do not scale to large and sparse real-world tensors with millions of rows and columns and ‘fibers.’ In this paper, we propose HATEN2, a distributed method for large-scale tensor decompositions that runs on the MAPREDUCE framework. Our careful design and implementation of HATEN2 dramati-

cally reduce the size of intermediate data and the number of jobs leading to achieve high scalability compared with the state-of-the-art method. Thanks to HATEN2, we analyze big real-world sparse tensors that cannot be handled by the current state of the art, and discover hidden concepts.

Keywords Tensor · Distributed computing · Big data · MapReduce · Hadoop

1 Introduction

Given historic records of millions of people calling each other, how can we identify telemarketers who make a lot of calls, but never receive ones? How can we detect suspicious attackers in a network packet transmission log? In general, how can we analyze large-scale real-world data with various attributes? Many real-world data (e.g., knowledge bases [1], web data [2], network traffic data [3], and many others [4–6]) with multiple attributes are represented as multi-dimensional arrays, called tensors. In analyzing a tensor, tensor decompositions are powerful tools in many data mining applications: correlation analysis on sensor streams [4], latent semantic indexing on DBLP publication data [7], multi-aspect forensics on network data [3], network discovery on fMRI data [6], to name a few. Using tensor decompositions, we find latent factors (or relations) within the data. These latent factors can be roughly and informally seen as soft clustering of the data. For example, decomposing a tensor constructed from the phone call history (with sender, receiver, date) into R latent factors corresponds to finding R clusters of senders that call a set of receivers on a specific date.

PARAFAC and Tucker are two widely used tensor decompositions. Since there is no single generalization of singular value decomposition (SVD) for a tensor, both PARAFAC

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and Tucker are considered as extensions of SVD to higher dimensions. PARAFAC is useful for decomposing a tensor into rank-one tensors which form the latent factors; Tucker is more suitable for compressing tensors and examining relations between the latent factors. Tucker is a more generalized version of PARAFAC, since the factors interact with all pairs of other factors.

Tensor decompositions have been extensively studied for various data mining tasks. However, most of tensor decomposition algorithms designed to deal with tensors fitting in main memory fail to decompose recent large-scale real-world tensors with millions and billions of rows, columns, and ‘fibers.’ The main challenge is to overcome the intermediate data explosion problem in distributed systems where the size of intermediate data exceeds the capacity of a single machine or even a cluster as the size of an input tensor gets larger. For example, if the size of the input tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ is $I = J = K = 1$ millions, the size of intermediate data of the n -mode product in Tucker would become 1 Exabyte ($=10^{18}$ bytes) with a straightforward implementation using distributed systems. Thus, we need to develop scalable and distributed tensor decomposition algorithms.

In this paper, we propose HATEN2 (which stands for HADOOP Tensor method for 2 decompositions), a scalable tensor decomposition suite for Tucker and PARAFAC decompositions on HADOOP [8], an open-source version of the MAPREDUCE framework [9]. HATEN2 provides both unconstrained and nonnegativity-constrained PARAFAC and Tucker decompositions: HATEN2-PARAFAC and HATEN2-Tucker for unconstrained version, and HATEN2-PARAFACNN and HATEN2-TuckerNN for nonnegativity-constrained version. Nonnegative tensor decomposition receives increasing attention because its result gives the benefit of easy interpretation, thanks to the nonnegativity in the factors. By carefully reordering operations and exploiting the sparsity of real-world tensors, HATEN2 solves the intermediate data explosion problem in distributed systems. Furthermore, HATEN2 significantly reduces the running time by integrating several redundant jobs. As a result, HATEN2 is able to analyze data that are several orders of magnitude larger than what the state of the art can handle. Applying HATEN2 to several real-world tensors, we discover malicious attackers in network traffic logs (with source IP, destination IP, port-number, timestamp), telemarketers in a phone call history (with sender, receiver, date), and interesting concepts in a knowledge base (with subject, object, relation).

Our main contributions are the following:

- *Algorithm* HATEN2 provides a unified framework, for unconstrained and nonnegativity-constrained Tucker and PARAFAC decompositions for sparse real-world tensors in distributed systems, which significantly reduce the intermediate data size and the running time.

- *Scalability* HATEN2 decomposes up to $100\times$ larger tensors without constraints as shown in Fig. 1 and up to $1000\times$ larger tensors with the nonnegativity constraint as shown in Fig. 8. Furthermore, HATEN2 scales up near linearly on the number of machines.
- *Discovery* By applying HATEN2, we discover interesting patterns on various real-world data—knowledge bases, network traffic logs, and phone call history—with millions of rows, columns, and fibers which were hard to analyze by existing methods.

The binary code and datasets are available at <http://datalab.snu.ac.kr/haten2>. The preliminary version of this work is described in [10]. In this work, we add two tensor decompositions (HaTen2-PARAFACNN and HaTen2-TuckerNN) for nonnegativity constraints and formulate them using our HaTen2 framework (Sect. 3.3). Furthermore, we present the additional discovery results on the three real-world tensors: concept discovery results on RDF knowledge base (Freebase-sampled), network traffic pattern discovery results on network traffic logs, and phone call pattern discovery results on phone call history (Phonecall) (Sects. 5.1, 5.2 and 5.3).

The rest of paper is organized as follows: Section 2 presents the preliminaries of the tensor and its decompositions. Section 3 describes our proposed method for the scalable tensor decompositions. After presenting the experimental results and discoveries in Sects. 4 and 5, we discuss related works in Sect. 6. Then we conclude in Sect. 7.

2 Preliminaries

In this section, we describe the preliminaries on tensor and its decompositions. Table 1 shows the definitions of symbols used in this paper. Matrices are denoted by boldface capitals (e.g., \mathbf{B}), and the r th row of the matrix \mathbf{B} is denoted by \mathbf{b}_r . Vectors are denoted by boldface lowercases (e.g., \mathbf{a}).

2.1 Tensor

Tensor Tensor is a multi-dimensional array. Each ‘dimension’ of a tensor is called *mode* or *way*. An N -mode or N -way tensor is denoted by $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$. $\text{bin}(\mathcal{X})$ denotes a function that converts nonzero elements in \mathcal{X} to 1. $\text{nnz}(\mathcal{X})$ means the number of nonzero elements of \mathcal{X} , and $\text{id}_x(\mathcal{X})$ means the set of indexes (e.g., (i, j, k) for 3-mode tensor \mathcal{X}) of nonzero elements in \mathcal{X} .

Fibers and slices A fiber is defined by fixing all but one index. In a 3-way tensor, it is denoted by $\mathcal{X}_{:jk}$, $\mathcal{X}_{i:k}$, and $\mathcal{X}_{ij:}$. A slice is defined by fixing all indices but two indices. In a 3-way tensor, it is denoted by $\mathcal{X}_{i:}$, $\mathcal{X}_{:j}$, and $\mathcal{X}_{:k}$.

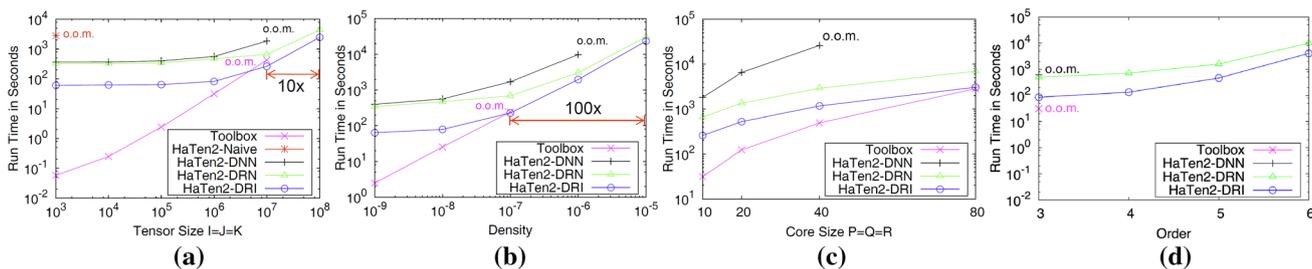


Fig. 1 Data scalability of our proposed HATen2-DRI compared to other methods, for Tucker decomposition. The datasets are explained in detail in Sect. 4.1. o.o.m.: out of memory. Density is defined to be the number of nonzeros divided by the number of all elements in a tensor.

Table 1 Table of symbols

| Symbol | Definition |
|---------------------------|---|
| \mathcal{X} | A tensor |
| $\mathbf{X}_{(n)}$ | Mode- n matricization of a tensor |
| a | A scalar (lowercase, italic letter) |
| \mathbf{a} | A column vector (lowercase, bold letter) |
| \mathbf{A} | A matrix (uppercase, bold letter) |
| R | Number of components |
| \circ | Outer product |
| \otimes | Kronecker product |
| \odot | Khatri–Rao product |
| $*$ | Hadamard product |
| \cdot | Standard product |
| $\bar{\times}_n$ | n -mode vector product |
| \times_n | n -mode matrix product |
| $\bar{*}_n$ | n -mode vector Hadamard product (Definition 1) |
| $*_n$ | n -mode matrix Hadamard product (Definition 5) |
| \mathbf{A}^T | Transpose of \mathbf{A} |
| $\ \mathbf{M}\ _F$ | Frobenius norm of \mathbf{M} |
| $\text{bin}(\mathcal{X})$ | Function that converts nonzero elements of \mathcal{X} to 1 |
| $\text{nnz}(\mathcal{X})$ | Number of nonzero elements in \mathcal{X} |
| $\text{idx}(\mathcal{X})$ | Set of indices $((i, j, k)$ or $(i, j, k, l))$ of nonzero elements in \mathcal{X} |
| I, J, K | Dimensions of each mode of input tensor \mathcal{X} |
| P, Q, R | Dimensions of each mode of core tensor \mathcal{G} |

Matricization of tensor The mode- n matricization of a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ is denoted by $\mathbf{X}_{(n)} \in \mathbb{R}^{I_n \times (\prod_{k \neq n} I_k)}$ and arranges the mode- n fibers to be the columns of the resulting matrix.

n -mode matrix product The n -mode matrix product of a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ with a matrix $\mathbf{U} \in \mathbb{R}^{J \times I_n}$ is denoted by $\mathcal{X} \times_n \mathbf{U}$ and is of size $I_1 \times \dots \times I_{n-1} \times J \times I_{n+1} \times \dots \times I_N$. It is defined by

$$(\mathcal{X} \times_n \mathbf{U})_{i_1 \dots i_{n-1} j i_{n+1} \dots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 i_2 \dots i_N} u_{j i_n}.$$

Note that our best method HATen2-DRI analyzes 10–100 larger data than the Tensor Toolbox. **a** Nonzeros and dimensionality, **b** density, **c** core, **d** order

n -mode vector product The n -mode vector product of a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ with a vector $\mathbf{v} \in \mathbb{R}^{I_n}$ is denoted by $\mathcal{X} \bar{\times}_n \mathbf{v}$ and is of size $I_1 \times \dots \times I_{n-1} \times I_{n+1} \times \dots \times I_N$. It is defined by

$$(\mathcal{X} \bar{\times}_n \mathbf{v})_{i_1 \dots i_{n-1} i_{n+1} \dots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 i_2 \dots i_N} v_{i_n}.$$

Kronecker product The Kronecker product of matrices $\mathbf{A} \in \mathbb{R}^{I \times J}$ and $\mathbf{B} \in \mathbb{R}^{K \times L}$ is denoted by $\mathbf{A} \otimes \mathbf{B}$. The result is a matrix of size $(IK) \times (JL)$ and defined by

$$\begin{aligned} \mathbf{A} \otimes \mathbf{B} &= \begin{bmatrix} \mathbf{a}_{11} \mathbf{B} & \mathbf{a}_{12} \mathbf{B} & \dots & \mathbf{a}_{1J} \mathbf{B} \\ \mathbf{a}_{21} \mathbf{B} & \mathbf{a}_{22} \mathbf{B} & \dots & \mathbf{a}_{2J} \mathbf{B} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{I1} \mathbf{B} & \mathbf{a}_{I2} \mathbf{B} & \dots & \mathbf{a}_{IJ} \mathbf{B} \end{bmatrix} \\ &= [\mathbf{a}_1 \otimes \mathbf{b}_1 \ \mathbf{a}_1 \otimes \mathbf{b}_2 \ \mathbf{a}_1 \otimes \mathbf{b}_3 \ \dots \ \mathbf{a}_J \otimes \mathbf{b}_{L-1} \ \mathbf{a}_J \otimes \mathbf{b}_L] \end{aligned}$$

Khatri–Rao product The Khatri–Rao product (or column-wise Kronecker product) $(\mathbf{A} \odot \mathbf{B})$, where \mathbf{A}, \mathbf{B} have the same number of columns, say R , is defined as:

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{A}(:, 1) \otimes \mathbf{B}(:, 1) \ \dots \ \mathbf{A}(:, R) \otimes \mathbf{B}(:, R)]$$

where $\mathbf{A}(:, r)$ is the r th column of \mathbf{A} . If \mathbf{A} is of size $I \times R$ and \mathbf{B} is of size $J \times R$ then $(\mathbf{A} \odot \mathbf{B})$ is of size $IJ \times R$.

Hadamard product The Hadamard product $\mathbf{A} * \mathbf{B}$ is the elementwise matrix product, where \mathbf{A} and \mathbf{B} have the same size $(I \times J)$, and is defined as:

$$\mathbf{A} * \mathbf{B} = \begin{bmatrix} \mathbf{a}_{11} \mathbf{b}_{11} & \mathbf{a}_{12} \mathbf{b}_{12} & \dots & \mathbf{a}_{1J} \mathbf{b}_{1J} \\ \mathbf{a}_{21} \mathbf{b}_{21} & \mathbf{a}_{22} \mathbf{b}_{22} & \dots & \mathbf{a}_{2J} \mathbf{b}_{2J} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{I1} \mathbf{b}_{I1} & \mathbf{a}_{I2} \mathbf{b}_{I2} & \dots & \mathbf{a}_{IJ} \mathbf{b}_{IJ} \end{bmatrix}$$

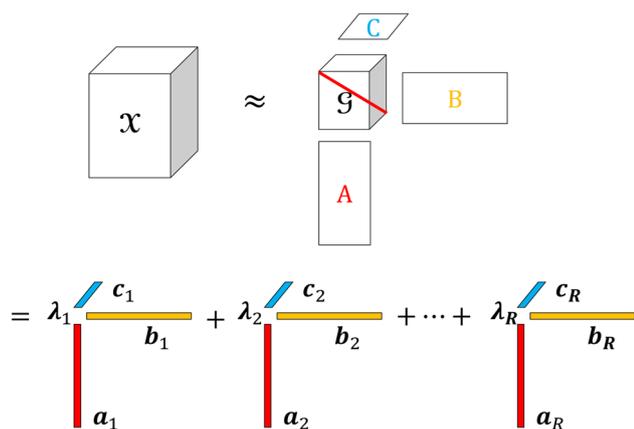


Fig. 2 Rank- R PARAFAC decomposition of a three-way tensor. The tensor \mathcal{X} is decomposed as three factor matrices \mathbf{A} , \mathbf{B} , and \mathbf{C}

2.2 Basic tensor decomposition

Tensor decomposition is a general tool for tensor analysis. Using tensor decomposition, we find latent factor or relations among data. In this paper, we focus on two major tensor decompositions, PARAFAC and Tucker.

2.2.1 PARAFAC decomposition

PARAFAC (parallel factors) decomposition [11], also called CANDECOMP (canonical decomposition), decomposes a tensor into a sum of rank-one tensors. There has been rich literature on algorithms for the PARAFAC decomposition, a concise summary thereof can be found in [12].

PARAFAC decomposition for 3-way tensor Given a 3-way tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ and rank R , PARAFAC decomposition factorizes the tensor into 3 factor matrices, \mathbf{A} , \mathbf{B} , and \mathbf{C} , as follows:

$$\mathcal{X} \approx [\lambda; \mathbf{A}, \mathbf{B}, \mathbf{C}] = \sum_{r=1}^R \lambda_r \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r$$

where R is a positive integer (typically between 10 and 100), λ is a weight vector, and $\mathbf{A} \in \mathbb{R}^{I \times R}$, $\mathbf{B} \in \mathbb{R}^{J \times R}$, and $\mathbf{C} \in \mathbb{R}^{K \times R}$ are the factor matrices. Figure 2 shows the 3-way PARAFAC tensor decomposition.

PARAFAC decomposition for N-way tensor Given an N -way tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ and rank R , PARAFAC decomposition factorizes the tensor into N factor matrices, $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \dots, \mathbf{A}^{(N)}$, as follows:

$$\mathcal{X} \approx [\lambda; \mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \dots, \mathbf{A}^{(N)}] = \sum_{r=1}^R \lambda_r \mathbf{a}_r^{(1)} \circ \mathbf{a}_r^{(2)} \circ \dots \circ \mathbf{a}_r^{(N)}$$

where R is a positive integer, λ is a weight vector, and $\mathbf{A}^{(1)} \in \mathbb{R}^{I_1 \times R}$, $\mathbf{A}^{(2)} \in \mathbb{R}^{I_2 \times R}$, ..., $\mathbf{A}^{(N)} \in \mathbb{R}^{I_N \times R}$ are the factor matrices.

PARAFAC-ALS Algorithm 1 shows the alternating least squares algorithm for 3-way PARAFAC decomposition where \dagger denotes the pseudo-inverse operation. The stopping criterion for Algorithm 1 is either one of the following: (1) the difference between the two least squares errors of consecutive iterations is smaller than a threshold or (2) the maximum number of iterations is exceeded.

Algorithm 1: 3-way PARAFAC-ALS.

Input: Tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$, rank R , maximum iterations T
Output: PARAFAC decomposition $\lambda \in \mathbb{R}^{R \times 1}$, $\mathbf{A} \in \mathbb{R}^{I \times R}$, $\mathbf{B} \in \mathbb{R}^{J \times R}$, $\mathbf{C} \in \mathbb{R}^{K \times R}$

- 1: Initialize $\mathbf{A}, \mathbf{B}, \mathbf{C}$;
- 2: **for** $t = 1, \dots, T$ **do**
- 3: $\mathbf{A} \leftarrow \mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B}) (\mathbf{C}^T \mathbf{C} * \mathbf{B}^T \mathbf{B})^\dagger$;
- 4: Normalize columns of \mathbf{A} (storing norms in vector λ);
- 5: $\mathbf{B} \leftarrow \mathbf{X}_{(2)} (\mathbf{C} \odot \mathbf{A}) (\mathbf{C}^T \mathbf{C} * \mathbf{A}^T \mathbf{A})^\dagger$;
- 6: Normalize columns of \mathbf{B} (storing norms in vector λ);
- 7: $\mathbf{C} \leftarrow \mathbf{X}_{(3)} (\mathbf{B} \odot \mathbf{A}) (\mathbf{B}^T \mathbf{B} * \mathbf{A}^T \mathbf{A})^\dagger$;
- 8: Normalize columns of \mathbf{C} (storing norms in vector λ);
- 9: **if** stopping criterion is met **then**
- 10: break for loop;
- 11: **end if**
- 12: **end for**
- 13: return $\lambda, \mathbf{A}, \mathbf{B}, \mathbf{C}$;

2.2.2 Tucker decomposition

In Tucker decomposition [13], called N-mode PCA or N-mode SVD, a tensor is decomposed into a core tensor and factor matrices of each mode. The factor matrices represent the principal components of each mode, and the core tensor represents the interactions between the different components. Tucker decomposition is a more generalized version of PARAFAC decomposition, since the factors interact with all pairs of other factors.

Tucker decomposition for 3-way tensor The 3-way tensor is decomposed as follows:

$$\begin{aligned} \mathcal{X} &\approx [\mathcal{G}; \mathbf{A}, \mathbf{B}, \mathbf{C}] = \mathcal{G} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C} \\ &= \sum_{p=1}^P \sum_{q=1}^Q \sum_{r=1}^R g_{pqr} \mathbf{a}_p \circ \mathbf{b}_q \circ \mathbf{c}_r \end{aligned}$$

where $\mathcal{G} \in \mathbb{R}^{P \times Q \times R}$ is the core tensor, and $\mathbf{A} \in \mathbb{R}^{I \times P}$, $\mathbf{B} \in \mathbb{R}^{J \times Q}$, and $\mathbf{C} \in \mathbb{R}^{K \times R}$ are the factor matrices. Figure 3 shows the Tucker decomposition of a 3-way tensor.

Tucker decomposition for N-way tensor The N -way tensor is decomposed as follows:

$$\begin{aligned} \mathcal{X} &\approx [\mathcal{G}; \mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \dots, \mathbf{A}^{(N)}] \\ &= \mathcal{G} \times_1 \mathbf{A}^{(1)} \times_2 \mathbf{A}^{(2)} \dots \times_N \mathbf{A}^{(N)} \end{aligned}$$

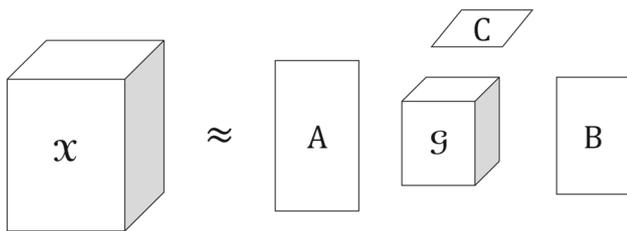


Fig. 3 Tucker decomposition of a three-way tensor. The tensor \mathcal{X} is decomposed as a core tensor \mathcal{G} , and three factor matrices \mathbf{A} , \mathbf{B} , and \mathbf{C}

where $\mathcal{G} \in \mathbb{R}^{J_1 \times J_2 \times \dots \times J_N}$ is the core tensor, and $\mathbf{A}^{(1)} \in \mathbb{R}^{I_1 \times J_1}$, $\mathbf{A}^{(2)} \in \mathbb{R}^{I_2 \times J_2}$, ..., and $\mathbf{A}^{(N)} \in \mathbb{R}^{I_N \times J_N}$ are the factor matrices.

Tucker-ALS Tucker-ALS algorithm uses an alternating least squares approach. For updating each factor, there are some approaches such as SVD, Bauer–Rutishauser, Gram–Schmidt and NIPALS [14].

Algorithm 2 shows the standard SVD-based algorithm for 3-way Tucker decomposition.

Algorithm 2: 3-way Tucker-ALS

- Input:** Tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$, desired core size: $P \times Q \times R$
Output: Core tensor $\mathcal{G} \in \mathbb{R}^{P \times Q \times R}$ and orthogonal factor matrices $\mathbf{A} \in \mathbb{R}^{I \times P}$, $\mathbf{B} \in \mathbb{R}^{J \times Q}$, and $\mathbf{C} \in \mathbb{R}^{K \times R}$
- 1: Initialize \mathbf{B} , \mathbf{C} ;
 - 2: **repeat**
 - 3: $\mathcal{Y} \leftarrow \mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$;
 - 4: $\mathbf{A} \leftarrow P$ leading left singular vectors of $\mathbf{Y}_{(1)}$;
 - 5: $\mathcal{Y} \leftarrow \mathcal{X} \times_1 \mathbf{A}^T \times_3 \mathbf{C}^T$;
 - 6: $\mathbf{B} \leftarrow Q$ leading left singular vectors of $\mathbf{Y}_{(2)}$;
 - 7: $\mathcal{Y} \leftarrow \mathcal{X} \times_1 \mathbf{A}^T \times_2 \mathbf{B}^T$;
 - 8: $\mathbf{C} \leftarrow R$ leading left singular vectors of $\mathbf{Y}_{(3)}$;
 - 9: $\mathcal{G} \leftarrow \mathcal{Y} \times_3 \mathbf{C}$;
 - 10: **until** $\|\mathcal{G}\|$ ceases to increase or the maximum number of outer iterations is exceeded.
-

2.3 Nonnegative tensor decomposition

Nonnegative tensor decomposition (NTD) is a tensor decomposition with the constraint enforcing all elements in the factors to be nonnegative. Since all elements have nonnegative values, the result of NTD is more interpretable. The most widely used method for nonnegative tensor decomposition or nonnegative matrix decomposition is the multiplicative update rule proposed by Lee and Seung [15]. They use the Euclidean distance and Kullback–Leibler divergence for the cost function. Lemma 1 [15] is the Euclidean distance version of the multiplicative update rule for nonnegative matrix decomposition.

Lemma 1 *When factoring the input matrix \mathbf{V} into \mathbf{WH} , the Euclidean distance $\|\mathbf{V} - \mathbf{WH}\|$ is nonincreasing under the update rules*

$$\mathbf{H}_{a\mu} \leftarrow \mathbf{H}_{a\mu} \frac{(\mathbf{W}^T \mathbf{V})_{a\mu}}{(\mathbf{W}^T \mathbf{W} \mathbf{H})_{a\mu}}, \mathbf{W}_{a\mu} \leftarrow \mathbf{W}_{a\mu} \frac{(\mathbf{V}^T \mathbf{H})_{a\mu}}{(\mathbf{W} \mathbf{H} \mathbf{H}^T)_{a\mu}}.$$

The Euclidean distance is invariant under these updates if and only if \mathbf{W} and \mathbf{H} are at a stationary point of the distance.

Since this rule is composed of only elementwise multiplication and division, if an initial matrix is nonnegative, then all interim matrices become nonnegative. The convergence of the multiplicative update rule is proved in the paper [15].

2.3.1 Nonnegative PARAFAC decomposition

The definition and algorithm of nonnegative PARAFAC decomposition for 3-way tensor are as follows:

Nonnegative PARAFAC decomposition for 3-way tensor Given a nonnegative tensor \mathcal{X} , the nonnegative PARAFAC decomposition solves

$$\min_{\mathbf{A}, \mathbf{B}, \mathbf{C}} \left\| \mathcal{X} - \sum_{r=1}^R \lambda_r \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r \right\|,$$

subject to $\mathbf{a}_r \geq 0$, $\mathbf{b}_r \geq 0$, $\mathbf{c}_r \geq 0$, where $\mathbf{A} \in \mathbb{R}_+^{I \times R}$, $\mathbf{B} \in \mathbb{R}_+^{J \times R}$, and $\mathbf{C} \in \mathbb{R}_+^{K \times R}$ are factor matrices of PARAFAC decomposition.

Nonnegative PARAFAC-ALS Algorithm 3 shows nonnegative PARAFAC-ALS algorithm for a 3-way tensor using multiplicative update rule [16]. The stopping criterion of Algorithm 3 is the same as that of Algorithm 1.

Algorithm 3: 3-way nonnegative PARAFAC-ALS

- Input:** Tensor $\mathcal{X} \in \mathbb{R}_+^{I \times J \times K}$, rank R , maximum iterations T .
Output: PARAFAC decomposition $\lambda \in \mathbb{R}^{R \times 1}$, $\mathbf{A} \in \mathbb{R}_+^{I \times R}$, $\mathbf{B} \in \mathbb{R}_+^{J \times R}$, $\mathbf{C} \in \mathbb{R}_+^{K \times R}$.
- 1: Initialize \mathbf{A} , \mathbf{B} , \mathbf{C} with nonnegative values;
 - 2: **for** $t = 1, \dots, T$ **do**
 - 3: $\mathbf{A} \leftarrow \mathbf{A} * \frac{\mathbf{X}_{(1)}(\mathbf{C} \circ \mathbf{B})}{\mathbf{A}(\mathbf{C}^T \mathbf{C} * \mathbf{B}^T \mathbf{B})}$;
 - 4: Normalize columns of \mathbf{A} (storing norms in vector λ);
 - 5: $\mathbf{B} \leftarrow \mathbf{B} * \frac{\mathbf{X}_{(2)}(\mathbf{C} \circ \mathbf{A})}{\mathbf{B}(\mathbf{C}^T \mathbf{C} * \mathbf{A}^T \mathbf{A})}$;
 - 6: Normalize columns of \mathbf{B} (storing norms in vector λ);
 - 7: $\mathbf{C} \leftarrow \mathbf{C} * \frac{\mathbf{X}_{(3)}(\mathbf{B} \circ \mathbf{A})}{\mathbf{C}(\mathbf{B}^T \mathbf{B} * \mathbf{A}^T \mathbf{A})}$;
 - 8: Normalize columns of \mathbf{C} (storing norms in vector λ);
 - 9: **if** stopping criterion is met **then**
 - 10: break for loop;
 - 11: **end if**
 - 12: **end for**
 - 13: return λ , \mathbf{A} , \mathbf{B} , \mathbf{C} ;
-

2.3.2 Nonnegative Tucker decomposition

The definition and algorithm of the nonnegative Tucker decomposition for 3-way tensor are as follows:

Nonnegative Tucker decomposition for 3-way tensor Given a nonnegative tensor \mathcal{X} , the nonnegative Tucker decomposition solves

$$\min_{\mathbf{A}, \mathbf{B}, \mathbf{C}} \|\mathcal{X} - \sum_{p=1}^P \sum_{q=1}^Q \sum_{r=1}^R g_{pqr} \mathbf{a}_p \circ \mathbf{b}_q \circ \mathbf{c}_r\|,$$

subject to $g_{pqr} \geq 0, \mathbf{a}_p \geq 0, \mathbf{b}_q \geq 0, \mathbf{c}_r \geq 0$, where $\mathcal{G} \in \mathbb{R}_+^{P \times Q \times R}$ is the core tensor, and $\mathbf{A} \in \mathbb{R}_+^{I \times P}, \mathbf{B} \in \mathbb{R}_+^{J \times Q}$, and $\mathbf{C} \in \mathbb{R}_+^{K \times R}$ are factor matrices of Tucker decomposition.

Nonnegative Tucker-ALS Kim and Choi extend multiplicative rule-based nonnegative PARAFAC decomposition to nonnegative Tucker decomposition. Algorithm 4 shows nonnegative Tucker-ALS algorithm for a 3-way tensor using multiplicative update rule [17].

Algorithm 4: 3-way nonnegative Tucker-ALS

Input: Tensor $\mathcal{X} \in \mathbb{R}_+^{I \times J \times K}$, desired core size: $P \times Q \times R$
Output: Core tensor $\mathcal{G} \in \mathbb{R}_+^{P \times Q \times R}$ and orthogonal factor matrices $\mathbf{A} \in \mathbb{R}_+^{I \times P}, \mathbf{B} \in \mathbb{R}_+^{J \times Q}$, and $\mathbf{C} \in \mathbb{R}_+^{K \times R}$
 1: Initialize \mathbf{B}, \mathbf{C} ;
 2: **repeat**
 3: $\mathbf{A} \leftarrow \mathbf{A} * \frac{(\mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T)_{(1)} \mathbf{G}_{(1)}^T}{\mathbf{A}(\mathcal{G} \times_1 \mathbf{B}^T \mathbf{B} \times_3 \mathbf{C}^T \mathbf{C})_{(1)} \mathbf{G}_{(1)}^T}$;
 4: Normalize columns of \mathbf{A} ;
 5: $\mathbf{B} \leftarrow \mathbf{B} * \frac{(\mathcal{X} \times_1 \mathbf{A}^T \times_3 \mathbf{C}^T)_{(2)} \mathbf{G}_{(2)}^T}{\mathbf{B}(\mathcal{G} \times_1 \mathbf{A}^T \mathbf{A} \times_3 \mathbf{C}^T \mathbf{C})_{(2)} \mathbf{G}_{(2)}^T}$;
 6: Normalize columns of \mathbf{B} ;
 7: $\mathbf{C} \leftarrow \mathbf{C} * \frac{(\mathcal{X} \times_1 \mathbf{A}^T \times_2 \mathbf{B}^T)_{(3)} \mathbf{G}_{(3)}^T}{\mathbf{C}(\mathcal{G} \times_1 \mathbf{A}^T \mathbf{A} \times_2 \mathbf{B}^T \mathbf{B})_{(3)} \mathbf{G}_{(3)}^T}$;
 8: Normalize columns of \mathbf{C} ;
 9: $\mathcal{G} \leftarrow \mathcal{G} * \frac{\mathcal{X} \times_1 \mathbf{A}^T \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T}{\mathcal{G} \times_1 \mathbf{A}^T \mathbf{A} \times_2 \mathbf{B}^T \mathbf{B} \times_3 \mathbf{C}^T \mathbf{C}}$;
 10: **until** $\|\mathcal{G}\|$ ceases to increase or the maximum number of outer iterations is exceeded.

3 Proposed method

In this section, we present HATEN2, our proposed distributed MAPREDUCE algorithms for large-scale tensor decompositions. Section 3.1 gives the main ideas of HATEN2, Sect. 3.2 describes details of HATEN2, and Sect. 3.3 extends HATEN2 for handling the nonnegativity constraints.

3.1 Overview

How can we design scalable and efficient PARAFAC/Tucker decomposition algorithms for very large tensors? The most challenging parts of those algorithms are n -mode matrix product $\mathcal{Y} \leftarrow \mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$ (lines 3, 5, and 7 of Algorithms 2 and 4) in Tucker-ALS and nonnegative Tucker-ALS, and Khatri–Rao product $\mathcal{Y} \leftarrow \mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B})$ (lines 3, 5, and 7 of Algorithms 1 and 3) in PARAFAC-ALS and nonnegative

PARAFAC-ALS. There are several challenges in designing efficient distributed algorithms for these operations.

- *Minimize intermediate data* During the computation, huge intermediate data are generated in the shuffle stage. How can we minimize the intermediate data?
- *Minimize disk accesses* How can we minimize the disk accesses to decrease the running time?
- *Minimize jobs* How can we minimize the number of MAPREDUCE jobs to decrease the running time?

Our main ideas to address the challenges are as follows:

- *Decoupling the steps in n -mode vector product* We decouple the multiplication and the addition steps in n -mode vector product by introducing a new operation called Hadamard-and-Merge which leads to decreasing the intermediate data size (Sect. 3.2.2).
- *Removing dependencies in sequential products* We remove dependencies by carefully reordering the computations and exploiting the sparsity of real-world tensors. It leads to further decreasing the intermediate data size (Sect. 3.2.3).
- *Integrating jobs by increasing memory usage* We integrate multiple MAPREDUCE jobs by increasing memory usage. The idea leads to minimizing the number of jobs and the disk accesses (Sect. 3.2.4).

Figure 4 shows the framework of our proposed HATEN2-DRI (or just HATEN2) method which contains all the above ideas. Note that although the computations for the two decompositions Tucker and PARAFAC are different, our HATEN2 unifies them into a general framework where the two methods differ only at the final merge step: HATEN2-Tucker uses *CrossMerge*, while HATEN2-PARAFAC uses *Pairwise-Merge* (see Sect. 3.2.4 for details). In the next subsection, we describe the three main ideas in detail.

3.2 Method details

In the following, we start with a naive method and improve the method gradually by adding several ideas one by one until we reach the final method HATEN2-DRI (or just HATEN2). Figure 5 and Table 2 summarize the differences between all methods. Tables 3 and 4 show the total costs of all the methods, in terms of the maximum intermediate data size, and the number of total MAPREDUCE jobs.

3.2.1 Naive method

The most naive method is the straightforward implementation of the idea in MET [5], the state-of-the-art single-

Fig. 4 General computational framework in HATEN2 for Tucker and PARAFAC decompositions ($Q = R$ in PARAFAC). Although the two decompositions are different, our HATEN2 unifies them into a general framework where the two methods differ only at the final merge step: *CrossMerge* for HATEN2-Tucker, and *PairwiseMerge* for HATEN2-PARAFAC (see Sect. 3.2.4 for details)

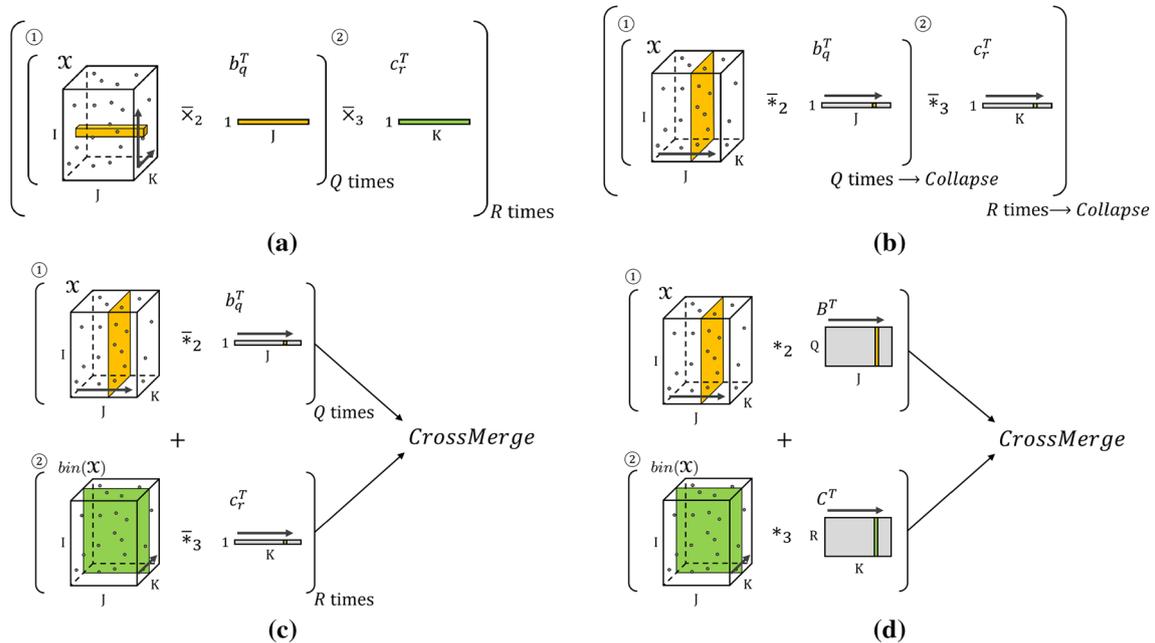
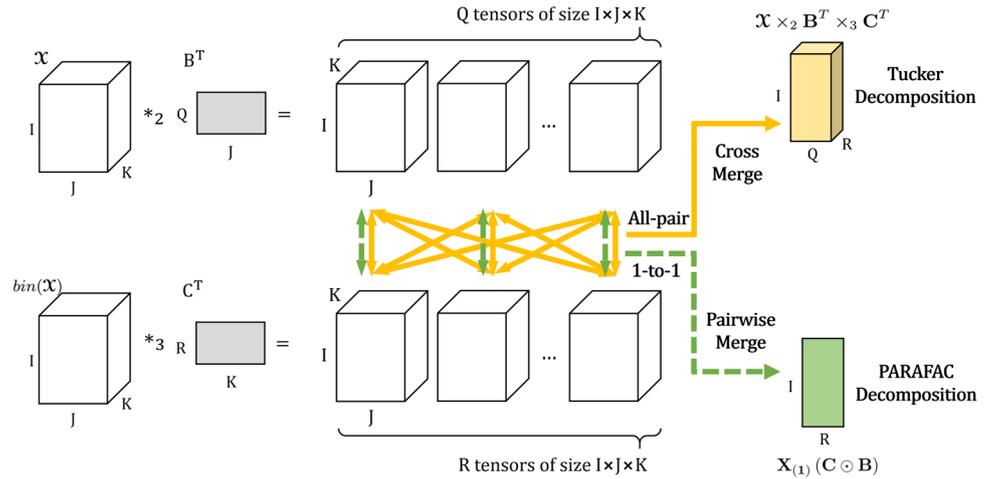


Fig. 5 Comparison of all HATEN2 variants for Tucker decomposition. Areas with the *same color* are sent to the same reducer in the MAPREDUCE jobs for n -mode (Hadamard) product

machine implementation which was adopted by Tensor Toolbox. The main idea is to perform each n -mode vector product separately. HATEN2-Tucker-Naive computes $\mathcal{T} = \mathcal{X} \times_2 \mathbf{B}^T$ first by performing $\mathcal{X} \bar{\times}_2 \mathbf{b}_q^T$ operation Q times and then computes $\mathcal{T} \times_3 \mathbf{C}^T$ by performing $\mathcal{T} \bar{\times}_3 \mathbf{c}_r^T$ operation R times, where \mathbf{b}_q and \mathbf{c}_r are the q th row of \mathbf{B} and the r th row of \mathbf{C} , respectively. Algorithm 5 shows HATEN2-Tucker-Naive method.

Similarly, HATEN2-PARAFAC-Naive computes $\mathcal{T}_r = \mathcal{X} \bar{\times}_2 \mathbf{b}_r^T$ first and then computes $\mathcal{Y}_r = \mathcal{T}_r \bar{\times}_3 \mathbf{c}_r^T$. It computes \mathcal{Y} by performing these operations R times. Algorithm 6 shows HATEN2-PARAFAC-Naive method.

Algorithm 5: HATEN2-Tucker-Naive for computing $\mathcal{Y} \leftarrow \mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$

Input: Tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ and factor matrices $\mathbf{B} \in \mathbb{R}^{J \times Q}$, $\mathbf{C} \in \mathbb{R}^{K \times R}$

Output: Tensor $\mathcal{Y} \in \mathbb{R}^{I \times Q \times R}$

- 1: for $q=1, \dots, Q$ do
- 2: $\mathcal{T}_q \leftarrow \mathcal{X} \bar{\times}_2 \mathbf{b}_q^T$;
- 3: end for
- 4: for $r=1, \dots, R$ do
- 5: $\mathcal{Y}_r \leftarrow \mathcal{T}_q \bar{\times}_3 \mathbf{c}_r^T$;
- 6: end for

MAPREDUCE algorithm The MAPREDUCE algorithm of n -mode vector product $\mathcal{X} \bar{\times}_2 \mathbf{b}_q^T$ in HATEN2-Naive is as follows:

Table 2 Comparison of all methods experimented

| Method | Distributed? | Decoupling the steps (D/N) (Sect. 3.2.2) | Remove dependencies (R/N) (Sect. 3.2.3) | Integrating jobs (I/N) (Sect. 3.2.4) |
|-----------------------------|--------------|---|--|---|
| Tensor toolbox | No | No | No | No |
| HATEN2-Naive | Yes | No | No | No |
| HATEN2-DNN | Yes | Yes | No | No |
| HATEN2-DRN | Yes | Yes | Yes | No |
| HATEN2-DRI (or just HATEN2) | Yes | Yes | Yes | Yes |

Our proposed and recommended method is HATEN2-DRI (or just HATEN2) which incorporates all the proposed ideas

Table 3 Summary of costs in all methods for computing $\mathcal{X} \times_2 \mathbf{B} \times_3 \mathbf{C}$ in Tucker decomposition. We replace $nnz(\mathcal{X} \times_2 \mathbf{B})$ with $nnz(\mathcal{X})Q$ according to the estimation of $nnz(\mathcal{X} \times_2 \mathbf{B})$ in Lemma 4

| Method | Max. intermediate data | Total jobs |
|---------------------|---------------------------|-------------|
| HATEN2-Tucker-Naive | $nnz(\mathcal{X}) + IJK$ | $Q + R$ |
| HATEN2-Tucker-DNN | $nnz(\mathcal{X})QR$ | $Q + R + 2$ |
| HATEN2-Tucker-DRN | $nnz(\mathcal{X})(Q + R)$ | $Q + R + 1$ |
| HATEN2-Tucker-DRI | $nnz(\mathcal{X})(Q + R)$ | 2 |

Table 4 Summary of costs in all methods for computing $\mathbf{X}_{(1)}(\mathbf{C} \odot \mathbf{B})$ in PARAFAC decomposition

| Method | Max. intermediate data | Total jobs |
|----------------------|--------------------------|------------|
| HATEN2-PARAFAC-Naive | $nnz(\mathcal{X}) + IJK$ | $2R$ |
| HATEN2-PARAFAC-DNN | $nnz(\mathcal{X}) + J$ | $4R$ |
| HATEN2-PARAFAC-DRN | $2nnz(\mathcal{X})R$ | $2R + 1$ |
| HATEN2-PARAFAC-DRI | $2nnz(\mathcal{X})R$ | 2 |

Algorithm 6: HATEN2-PARAFAC-Naive for computing $\mathcal{Y} \leftarrow \mathbf{X}_{(1)}(\mathbf{C} \odot \mathbf{B})$

Input: Tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ and factor matrices $\mathbf{B} \in \mathbb{R}^{J \times R}$, $\mathbf{C} \in \mathbb{R}^{K \times R}$

Output: Tensor $\mathcal{Y} \in \mathbb{R}^{I \times R}$

- 1: **for** $r=1, \dots, R$ **do**
- 2: $\mathcal{J}_r \leftarrow \mathcal{X} \bar{\times}_2 \mathbf{b}_r^T$;
- 3: $\mathcal{Y}_r \leftarrow \mathcal{J}_r \bar{\times}_3 \mathbf{c}_r^T$;
- 4: **end for**

<Naive: $\mathcal{X} \bar{\times}_2 \mathbf{b}_q^T$ >

- **MAP:** map $\langle i, j, k, \mathcal{X}(i, j, k) \rangle$ on $(iK + k)$, such that tuples with the same key are shuffled to the same reducer in the form of $\langle \text{key: } (iK + k), \text{ values: } \{(j, \mathcal{X}(i, j, k)) | \forall (i, j, k) \in \text{idx}(\mathcal{X}_{i:k})\} \rangle$, and send \mathbf{b}_q to the all the reducers in the form of $\langle \text{key: } (iK + k) | \forall (i, k), \text{ values: } \{(j, \mathbf{b}_q(j)) | \forall j \in \text{idx}(\mathbf{b}_q)\} \rangle$.
- **REDUCE:** take $\langle \text{key: } (iK + k), \text{ values: } \{(j, \mathbf{b}_q(j)) | \forall j \in \text{idx}(\mathbf{b}_q)\}, \{(j, \mathcal{X}(i, j, k)) | \forall (i, j, k) \in \text{idx}(\mathcal{X}_{i:k})\} \rangle$, and emit $\langle i, k, \sum_{j=1}^J \mathcal{X}(i, j, k) \mathbf{b}_q(j) \rangle$.

The reducer processing the key $iK + k$ receives the nonzero elements of $\mathcal{X}_{i:k}$ and \mathbf{b}_q . Then it performs the inner product of the two vectors and outputs an element of the result tensor \mathcal{J}_q . $\mathcal{J} \bar{\times}_3 \mathbf{c}_r^T$ operation is handled in the same manner.

Although simple, this naive implementation has too much overhead because (1) the vector \mathbf{b}_q^T is copied IK times which eventually generates too much intermediate data ($nnz(\mathcal{X}) + IJK$) and (2) the vector \mathbf{b}_q^T might not fit in the memory of a machine when J is very large. How can we improve this naive method? In the following three subsections, we incrementally improve the naive method.

3.2.2 Decoupling the steps in n -mode vector product

The first idea to improve the naive method is to make the n -mode vector product $\bar{\times}_n$ scalable. As we saw in the previous subsection, the naive algorithm which broadcasts the vector \mathbf{b}_q^T has too much overhead. Our idea, called Hadamard-and-Merge, is to decouple the product into two steps: The Hadamard product step where the element of the vector is multiplied with the corresponding element of the tensor, and the merge step where the multiplied values are summed. Hadamard-and-Merge operation comprises the two following operations: n -mode vector Hadamard product and Collapse.

Definition 1 (n -mode vector Hadamard product) The n -mode vector Hadamard product of a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ and a vector $\mathbf{v} \in \mathbb{R}^{I_n}$ is denoted by $\mathcal{X} \bar{\times}_n \mathbf{v}$ and is of size $I_1 \times I_2 \times \dots \times I_N$. It is defined by

$$(\mathcal{X} \bar{*}_n \mathbf{v})_{i_1 \dots i_n \dots i_N} = x_{i_1 \dots i_n \dots i_N} v_{i_n}$$

Definition 2 ($Collapse(\mathcal{X})_n$) The $Collapse$ operation of a tensor $\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ on mode n is denoted by $Collapse(\mathcal{X})_n$ and is of size $I_1 \times \dots \times I_{(n-1)} \times I_{(n+1)} \times \dots \times I_N$. It is defined by

$$(Collapse(\mathcal{X})_n)_{i_1 \dots i_{n-1} i_{n+1} \dots i_N} = \sum_{i_n=1}^{I_n} x_{i_1 \dots i_n \dots i_N}$$

Intuitively, the n -mode vector Hadamard product is a generalization of Hadamard product of two vectors. The $Collapse(\mathcal{X})_n$ operation sums up all the values of a tensor \mathcal{X} across the mode n . With these definitions, HATEN2-DNN expresses the original n -mode vector product $\mathcal{X} \bar{*}_2 \mathbf{b}_q^T$ by $Collapse(\mathcal{X} \bar{*}_2 \mathbf{b}_q^T)_2$. By decoupling the n -mode vector product into two steps, HATEN2-DNN greatly decreases the intermediate data size of HATEN2-Naive from $nnz(\mathcal{X}) + IJK$ to $nnz(\mathcal{X})QR$ for Tucker, and from $nnz(\mathcal{X}) + IJK$ to $nnz(\mathcal{X}) + J$ for PARAFAC. Algorithms 7 and 8 show HATEN2-DNN for Tucker and PARAFAC decompositions, respectively. In HATEN2-Tucker-DNN, we compute $\mathcal{T} = \mathcal{X} \times_2 \mathbf{B}^T \in \mathbb{R}^{I \times Q \times K}$ by iteratively performing $\mathcal{T}'_q = \mathcal{X} \bar{*}_2 \mathbf{b}_q^T$ for Q times and then merging them using the operation $Collapse(\mathcal{T}')_2$. $\mathcal{T} \times_3 \mathbf{C}^T$ operation is handled in the same manner. In HATEN2-PARAFAC-DNN, $Collapse$ is applied right after the individual n -mode vector Hadamard product.

Algorithm 7: HATEN2-Tucker-DNN for computing $\mathcal{Y} \leftarrow \mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$

Input: Tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ and factor matrices $\mathbf{B} \in \mathbb{R}^{J \times Q}$, $\mathbf{C} \in \mathbb{R}^{K \times R}$

Output: Tensor $\mathcal{Y} \in \mathbb{R}^{I \times Q \times R}$

- 1: **for** $q=1, \dots, Q$ **do**
 - 2: $\mathcal{T}'_q \leftarrow \mathcal{X} \bar{*}_2 \mathbf{b}_q^T$;
 - 3: **end for**
 - 4: $\mathcal{T} \leftarrow Collapse(\mathcal{T}')_2$;
 - 5: **for** $r=1, \dots, R$ **do**
 - 6: $\mathcal{Y}'_r \leftarrow \mathcal{T} \bar{*}_3 \mathbf{c}_r^T$;
 - 7: **end for**
 - 8: $\mathcal{Y} \leftarrow Collapse(\mathcal{Y}')_3$;
-

MAPREDUCE algorithm The MAPREDUCE algorithms of n -mode vector Hadamard product and $Collapse(\mathcal{X})_n$ in HATEN2-DNN are expressed as follows:

$$\langle \mathcal{X} \bar{*}_2 \mathbf{b}_q^T \rangle$$

- **MAP:** map $\langle i, j, k, \mathcal{X}(i, j, k) \rangle$ on j , and $\langle j, q, \mathbf{b}_q(j) \rangle$ on j such that tuples with the same key are shuffled to the same reducer in the form of $\langle \text{key: } j, \text{ values: } (q, \mathbf{b}_q(j)), \{(i, k, \mathcal{X}(i, j, k)) | \forall (i, k) \in idx(\mathcal{X}_{i:k})\} \rangle$.

Algorithm 8: HATEN2-PARAFAC-DNN for computing $\mathcal{Y} \leftarrow \mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B})$

Input: Tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ and factor matrices $\mathbf{B} \in \mathbb{R}^{J \times R}$, $\mathbf{C} \in \mathbb{R}^{K \times R}$

Output: Tensor $\mathcal{Y} \in \mathbb{R}^{I \times R}$

- 1: **for** $r=1, \dots, R$ **do**
 - 2: $\mathcal{T}'_r \leftarrow \mathcal{X} \bar{*}_2 \mathbf{b}_r^T$;
 - 3: $\mathcal{T}_r \leftarrow Collapse(\mathcal{T}'_r)_2$;
 - 4: $\mathcal{Y}'_r \leftarrow \mathcal{T}_r \bar{*}_3 \mathbf{c}_r^T$;
 - 5: $\mathcal{Y}_r \leftarrow Collapse(\mathcal{Y}'_r)_3$;
 - 6: **end for**
-

- **REDUCE:** take $\langle \text{key: } j, \text{ values: } (q, \mathbf{b}_q(j)), \{(i, k, \mathcal{X}(i, j, k)) | \forall (i, k) \in idx(\mathcal{X}_{i:k})\} \rangle$ and emit $\langle i, j, k, q, \mathcal{X}(i, j, k) \mathbf{b}_q(j) \rangle$ for each $(i, k) \in idx(\mathcal{X}_{i:k})$.

$$\langle Collapse(\mathcal{T}')_2 \rangle$$

- **MAP:** map $\langle i, j, k, q, \mathcal{X}(i, j, k) \mathbf{b}_q(j, q) \rangle$ on $(iK + k)$ such that tuples with the same key are shuffled to the same reducer in the form of $\langle \text{key: } (iK + k), \text{ values: } \{(q, \mathcal{X}(i, j, k) \mathbf{b}_q(j, q)) | \forall (i, j, k, q) \in idx(\mathcal{T}_{i:k})\} \rangle$.
- **REDUCE:** take $\langle \text{key: } (iK + k), \text{ values: } \{(q, \mathcal{X}(i, j, k) \mathbf{b}_q(j, q)) | \forall (i, j, k, q) \in idx(\mathcal{T}_{i:k})\} \rangle$ and emit $\langle i, q, k, \sum_j \mathcal{X}(i, j, k) \mathbf{b}_q(j, q) \rangle$ for each $(i, j, k, q) \in idx(\mathcal{T}_{i:k})$.

In n -mode vector Hadamard product, the mappers send $nnz(\mathcal{X}_{:j})$ of j th slice and an element of \mathbf{b}_q to reducers using j as the key. The reducers multiply the vector element with the tensor elements. In $Collapse$ operation, mappers send $nnz(\mathcal{X}_{i:k})Q$ elements to reducers using $iK + k$ as the key. The reducers aggregate the values.

3.2.3 Removing dependencies in sequential products

The previous two methods HATEN2-Naive and HATEN2-DNN have a significant problem: they have dependencies in their computation sequences. In Tucker decomposition, to compute $\mathcal{Y} \leftarrow \mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$, both of the previous methods first compute $\mathcal{T} = \mathcal{X} \times_2 \mathbf{B}^T$ by multiplying \mathcal{X} and columns of B , and then multiply \mathcal{T} with the columns of C . That is, the second step cannot be initiated until the first step is finished. Similarly, in PARAFAC decomposition, computing $\mathcal{Y} \leftarrow \mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B})$ has a dependency: \mathcal{X} is multiplied with \mathbf{b}^T first, and the result is multiplied with \mathbf{c}^T . These dependencies in the computation have the following problems.

- Too large intermediate data: in Tucker decomposition, the number $nnz(\mathcal{T})$ of nonzero elements in $\mathcal{T} = \mathcal{X} \times_2 \mathbf{B}^T$ is estimated to be $nnz(\mathcal{X})Q$ for a sparse tensor \mathcal{X} , as described in Lemma 4. Thus, multiplying \mathcal{T} with \mathbf{C}^T would require intermediate data of size $nnz(\mathcal{X})QR$ which is prohibitively large.

- Too many MAPREDUCE jobs: in PARAFAC decomposition, HATEN2-PARAFAC-DNN requires $4R$ MAPREDUCE jobs, since the first multiplication with \mathbf{b}^T , and the second multiplication with \mathbf{c}^T are performed in sequence.

Our idea to solve the problems is to remove the dependencies by carefully reordering the computations, and exploiting the sparsity of real-world tensors. Before describing the details, we introduce two new operations *CrossMerge* and *PairwiseMerge* as follows:

Definition 3 (*CrossMerge*) The *CrossMerge* operation of $N - 1$ tensors $\mathbf{X}_1 \in \mathbb{R}^{I_1 \times \dots \times I_N \times J_1}, \dots, \mathbf{X}_{n-1}, \mathbf{X}_{n+1}, \dots, \mathbf{X}_N \in \mathbb{R}^{I_1 \times \dots \times I_N \times J_N}$ on the mode n is denoted by $CrossMerge(\mathbf{X}_1, \dots, \mathbf{X}_{n-1}, \mathbf{X}_{n+1}, \dots, \mathbf{X}_N)_{(n)}$ and is of size $\mathbb{R}^{I_n \times J_1 \times \dots \times J_N}$. It is defined by

$$(CrossMerge(\mathbf{X}_1, \dots, \mathbf{X}_{n-1}, \mathbf{X}_{n+1}, \dots, \mathbf{X}_N)_{(n)})_{i_n j_1 \dots j_N} = \sum_{(i_1, \dots, i_{n-1}, i_{n+1}, \dots, i_N) = (1, \dots, 1)} \mathbf{X}_1(i_1, \dots, i_N, j_1) \times \dots \times \mathbf{X}_m(i_1, \dots, i_N, j_N),$$

for all $j_i = 1, \dots, J_i$ where $i \neq n$.

Definition 4 (*PairwiseMerge*) The *PairwiseMerge* operation of $N - 1$ tensors $\mathbf{X}_1, \dots, \mathbf{X}_{n-1}, \mathbf{X}_{n+1}, \dots, \mathbf{X}_N \in \mathbb{R}^{I_1 \times \dots \times I_N \times J}$ on the mode n is denoted by $PairwiseMerge(\mathbf{X}_1, \dots, \mathbf{X}_{n-1}, \mathbf{X}_{n+1}, \dots, \mathbf{X}_N)_{(n)}$ and is of size $\mathbb{R}^{I_n \times J}$. It is defined by

$$(PairwiseMerge(\mathbf{X}_1, \dots, \mathbf{X}_{n-1}, \mathbf{X}_{n+1}, \dots, \mathbf{X}_N)_{(n)})_{i_n j} = \sum_{(i_1, \dots, i_{n-1}, i_{n+1}, \dots, i_N) = (1, \dots, 1)} \mathbf{X}_1(i_1, \dots, i_N, j) \times \dots \times \mathbf{X}_m(i_1, \dots, i_N, j),$$

for all $j = 1, \dots, J$.

Our crucial observation is that these two operations can be used for removing the dependencies in the computations of $\mathbf{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$ and $\mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B})$, respectively, as shown in the following lemmas.

Lemma 2 (*CrossMerge*) Given $\mathbf{X} \in \mathbb{R}^{I \times J \times K}$, $\mathbf{B} \in \mathbb{R}^{J \times Q}$, and $\mathbf{C} \in \mathbb{R}^{K \times R}$,

$$\mathbf{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T \Leftrightarrow CrossMerge(\mathcal{F}', \mathcal{F}'')_{(1)}$$

where $\mathcal{F}' \in \mathbb{R}^{I \times J \times K \times Q}$ is a tensor whose q th subtensor $\mathcal{F}'_{\dots q}$ is given by $\mathbf{X} \bar{\ast}_2 \mathbf{b}_q^T$, and $\mathcal{F}'' \in \mathbb{R}^{I \times J \times K \times R}$ is a tensor whose r th subtensor $\mathcal{F}''_{\dots r}$ is given by $bin(\mathbf{X}) \bar{\ast}_3 \mathbf{c}_r^T$.

Proof The (i, q, k) th element $\mathcal{M}_{i q k}$ of $\mathcal{M} = \mathbf{X} \times_2 \mathbf{B}^T$ is given by

$$\mathcal{M}_{i q k} = \sum_{j=1}^J \mathbf{X}(i, j, k) \mathbf{B}(j, q).$$

Then the (i, q, r) th element of $(\mathbf{X} \times_2 \mathbf{B}^T) \times_3 \mathbf{C}^T$ is

$$\begin{aligned} & \sum_{k=1}^K \mathcal{M}(i, q, k) \mathbf{C}(k, r) \\ &= \sum_{k=1}^K \left(\sum_{j=1}^J \mathbf{X}(i, j, k) \mathbf{B}(j, q) \right) \mathbf{C}(k, r) \\ &= \sum_{(j,k)=(1,1)}^{(J,K)} \mathbf{X}(i, j, k) \mathbf{B}(j, q) \mathbf{C}(k, r) \end{aligned} \tag{1}$$

The (i, j, k, q) th element of subtensor $\mathcal{F}'_{\dots q}$ is given by $\mathbf{X}(i, j, k) \mathbf{b}_q^T(j)$, and (i, j, k, r) th element of subtensor $\mathcal{F}''_{\dots r}$ is given by $(bin(\mathbf{X})(i, j, k)) \mathbf{c}_r^T(k)$. Therefore, the (i, j, k, q) th element of \mathcal{F}' is

$$\mathcal{F}'_{i j k q} = \mathbf{X}(i, j, k) \mathbf{B}(j, q),$$

and the (i, j, k, r) th element of \mathcal{F}'' is

$$\mathcal{F}''_{i j k r} = (bin(\mathbf{X})(i, j, k)) \mathbf{C}(k, r).$$

The (i, q, r) th element of $CrossMerge(\mathcal{F}', \mathcal{F}'')_{(1)}$ is

$$\begin{aligned} & \sum_{(j,k)=(1,1)}^{(J,K)} \mathcal{F}'(i, j, k, q) \mathcal{F}''(i, j, k, r) \\ &= \sum_{(j,k)=(1,1)}^{(J,K)} \mathbf{X}(i, j, k) \mathbf{B}(j, q) (bin(\mathbf{X})(i, j, k)) \mathbf{C}(k, r) \\ &= \sum_{(j,k)=(1,1)}^{(J,K)} \mathbf{X}(i, j, k) \mathbf{B}(j, q) \mathbf{C}(k, r) \end{aligned} \tag{2}$$

where the last equality uses the fact $\mathbf{X}(i, j, k) \times (bin(\mathbf{X})(i, j, k)) = \mathbf{X}(i, j, k)$. The Eq. (1) for (i, q, r) th element of $\mathbf{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$ is exactly the same as the Eq. (2) for (i, q, r) th element of $CrossMerge(\mathcal{F}', \mathcal{F}'')_{(1)}$.

Lemma 3 (*PairwiseMerge*) Given $\mathbf{X} \in \mathbb{R}^{I \times J \times K}$, $\mathbf{B} \in \mathbb{R}^{J \times R}$, and $\mathbf{C} \in \mathbb{R}^{K \times R}$,

$$\mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B}) \Leftrightarrow PairwiseMerge(\mathcal{F}', \mathcal{F}'')_{(1)}$$

where $\mathcal{F}' \in \mathbb{R}^{I \times J \times K \times R}$ is a tensor whose r th subtensor $\mathcal{F}'_{\dots r}$ is given by $\mathbf{X} \bar{\ast}_2 \mathbf{b}_r^T$, and $\mathcal{F}'' \in \mathbb{R}^{I \times J \times K \times R}$ is a tensor whose r th subtensor $\mathcal{F}''_{\dots r}$ is given by $bin(\mathbf{X}) \bar{\ast}_3 \mathbf{c}_r^T$.

Proof The (i, r) th element of $\mathbf{M} = \mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B})$ is defined by

$$\mathbf{M}_{i r} = \sum_{(j,k)=(1,1)}^{(J,K)} \mathbf{X}(i, j, k) \mathbf{B}(j, r) \mathbf{C}(k, r) \tag{3}$$

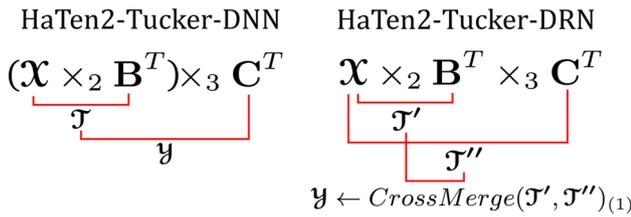


Fig. 6 Comparison of HATEN2-Tucker-DNN and HATEN2-Tucker-DRN

The (i, j, k) th element of subtensor $\mathcal{F}'_{\dots r}$ is given by $\mathcal{X}(i, j, k)\mathbf{b}_r^T(j)$, and the (i, j, k) th element of subtensor $\mathcal{F}''_{\dots r}$ is given by $(\text{bin}(\mathcal{X})(i, j, k))\mathbf{c}_r^T(k)$. Therefore, the (i, j, k, r) th element of \mathcal{F}' is

$$\mathcal{F}'_{ijklr} = \mathcal{X}(i, j, k)\mathbf{B}(j, r),$$

and the (i, j, k, r) th element of \mathcal{F}'' is

$$\mathcal{F}''_{ijklr} = (\text{bin}(\mathcal{X})(i, j, k))\mathbf{C}(k, r).$$

The (i, r) th element of $\text{PairwiseMerge}(\mathcal{F}', \mathcal{F}'')_{(1)}$ is

$$\begin{aligned} & \sum_{(j,k)=(1,1)}^{(J,K)} \mathcal{F}'(i, j, k, r)\mathcal{F}''(i, j, k, r). \\ &= \sum_{(j,k)=(1,1)}^{(J,K)} \mathcal{X}(i, j, k)\mathbf{B}(j, r)(\text{bin}(\mathcal{X})(i, j, k))\mathbf{C}(k, r) \\ &= \sum_{(j,k)=(1,1)}^{(J,K)} \mathcal{X}(i, j, k)\mathbf{B}(j, r)\mathbf{C}(k, r) \cdots (2). \end{aligned} \quad (4)$$

where the last equality uses the fact $\mathcal{X}(i, j, k) \times (\text{bin}(\mathcal{X})(i, j, k)) = \mathcal{X}(i, j, k)$. The Eq. (3) for (i, r) th element of $\mathbf{X}_{(1)}(\mathbf{C} \odot \mathbf{B})$ is exactly the same as the Eq. (4) for (i, r) th element of $\text{PairwiseMerge}(\mathcal{F}', \mathcal{F}'')_{(1)}$. \square

Using these two operations, HATEN2-DRN-Tucker, shown in Algorithm 9, computes \mathcal{F}' and \mathcal{F}'' first, and then merges the result. Figure 6 illustrates the difference of HATEN2-Tucker-DRN and HATEN2-Tucker-DNN. Note that both \mathcal{F}' and \mathcal{F}'' are sparse if the input tensor \mathcal{X} is sparse, which is true in most real-world tensors. Thus, HATEN2-Tucker-DRN further decreases the intermediate data size of HATEN2-DNN from $\text{nnz}(\mathcal{X})QR$ to $\text{nnz}(\mathcal{X})(Q + R)$. We want to emphasize that this decrease of the intermediate data size comes from the sparsity of real-world tensors where $\text{nnz}(\mathcal{X}) \sim I$; if the input tensor is a full tensor (which is not realistic), the intermediate data size of HATEN2-DNN becomes $\text{nnz}(\mathcal{X})Q$ which is smaller than that of HATEN2-DRN. Also, note that the removal of dependency in HATEN2-DRN enables computing \mathcal{F}' and \mathcal{F}'' in parallel; the

idea is reflected in HATEN2-DRI (see Sect. 3.2.4 for details). For PARAFAC decomposition, HATEN2-PARAFAC-DRN, shown in Algorithm 10, decreases the number of MAPREDUCE jobs of HATEN2-PARAFAC-DNN from $4R$ to $2R + 1$.

Algorithm 9: HATEN2-Tucker-DRN for computing $\mathcal{Y} \leftarrow \mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$

```

Input: Tensor  $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$  and factor matrices  $\mathbf{B} \in \mathbb{R}^{J \times Q}$ ,
 $\mathbf{C} \in \mathbb{R}^{K \times R}$ 
Output: Tensor  $\mathcal{Y} \in \mathbb{R}^{I \times Q \times R}$ 
1: for  $q=1, \dots, Q$  do
2:    $\mathcal{F}'_q \leftarrow \mathcal{X} \bar{\ast}_2 \mathbf{b}_q^T$ ;
3: end for
4: for  $r=1, \dots, R$  do
5:    $\mathcal{F}''_r \leftarrow \text{bin}(\mathcal{X}) \bar{\ast}_3 \mathbf{c}_r^T$ ;
6: end for
7:  $\mathcal{Y} \leftarrow \text{CrossMerge}(\mathcal{F}', \mathcal{F}'')_{(1)}$ ;
    
```

Algorithm 10: HATEN2-PARAFAC-DRN for computing $\mathcal{Y} \leftarrow \mathbf{X}_{(1)}(\mathbf{C} \odot \mathbf{B})$

```

Input: Tensor  $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$  and factor matrices  $\mathbf{B} \in \mathbb{R}^{J \times R}$ ,
 $\mathbf{C} \in \mathbb{R}^{K \times R}$ 
Output: Tensor  $\mathcal{Y} \in \mathbb{R}^{I \times R}$ 
1: for  $r=1, \dots, R$  do
2:    $\mathcal{F}'_r \leftarrow \mathcal{X} \bar{\ast}_2 \mathbf{b}_r^T$ ;
3: end for
4: for  $r=1, \dots, R$  do
5:    $\mathcal{F}''_r \leftarrow \text{bin}(\mathcal{X}) \bar{\ast}_3 \mathbf{c}_r^T$ ;
6: end for
7:  $\mathcal{Y} \leftarrow \text{PairwiseMerge}(\mathcal{F}', \mathcal{F}'')_{(1)}$ ;
    
```

MAPREDUCE algorithm The MAPREDUCE algorithms of $\text{CrossMerge}(\mathcal{F}', \mathcal{F}'')_{(1)}$ and $\text{PairwiseMerge}(\mathcal{F}', \mathcal{F}'')_{(1)}$ operations are as follows:

- $\langle \text{CrossMerge}(\mathcal{F}', \mathcal{F}'')_{(1)} \rangle$
- **MAP:** map $\langle i, j, k, q, \mathcal{X}(i, j, k)\mathbf{b}_q(j) \rangle$ on $(i, rQ + q)$ for all $r = 1, \dots, R$, and $\langle i, j, k, r, \mathbf{c}_r(k) \rangle$ on $(i, rQ + q)$ for all $q = 1, \dots, Q$ such that tuples with the same key are shuffled to the same reducer in the form of $\langle \text{key}: (i, rQ + q), \text{values}: \{ (j, k, \mathcal{X}(i, j, k)\mathbf{b}_q(j)) | \forall (i, j, k) \in \text{idx}(\mathcal{X}_{i::}) \}, \{ (j, k, r, \mathbf{c}_r(k)) | \forall (i, j, k) \in \text{idx}(\mathcal{X}_{i::}) \} \rangle$.
- **REDUCE:** take $\langle \text{key}: (i, rQ + q), \text{values}: \{ (j, k, \mathcal{X}(i, j, k)\mathbf{b}_q(j)) | \forall (i, j, k) \in \text{idx}(\mathcal{X}_{i::}) \}, \{ (j, k, r, \mathbf{c}_r(k)) | \forall (i, j, k) \in \text{idx}(\mathcal{X}_{i::}) \} \rangle$ and emit $\langle \{ i, q, r, \sum_{j,k} \mathcal{X}(i, j, k)\mathbf{b}_q(j)\mathbf{c}_r(k) \text{ for all } q = 1, \dots, Q, r = 1, \dots, R \} \rangle$ for each $(i, j, k) \in \text{idx}(\mathcal{X}_{i::})$.

$\langle \text{PairwiseMerge}(\mathcal{F}', \mathcal{F}'')_{(1)} \rangle$

- **MAP:** map $\langle i, j, k, r, \mathcal{X}(i, j, k)\mathbf{b}_r(j) \rangle$ on (i, r) , and $\langle i, j, k, r, \mathbf{c}_r(k) \rangle$ on (i, r) such that tuples with the same key are shuffled to the same reducer in the form of

- <key: (i, r) , values: $\{(j, k, \mathcal{X}(i, j, k)\mathbf{b}_r(j)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{i:,:})\}, \{(j, k, \mathbf{c}_r(k)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{i:,:})\}$ >.
- **REDUCE**: take <key: (i, r) , values: $\{(j, k, \mathcal{X}(i, j, k)\mathbf{b}_r(j)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{i:,:})\}, \{(j, k, \mathbf{c}_r(k)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{i:,:})\}$ > and emit $\langle i, r, \sum_{j,k} \mathcal{X}(i, j, k)\mathbf{b}_r(j)\mathbf{c}_r(k) \rangle$ for all $r = 1, \dots, R$ for each $(i, j, k) \in \text{idx}(\mathcal{X}_{i:,:})$.

3.2.4 Integrating jobs by increasing memory usage

Although HATEN2-DRN decreased the intermediate data size and the number of MAPREDUCE jobs, the number of jobs is still significant: it is $Q + R + 1$ for HATEN2-Tucker-DRN and $2R + 1$ for HATEN2-PARAFAC-DRN. In this subsection, we propose HATEN2-DRI to further decrease the number of jobs to 2, thereby decreasing the disk accesses and the running time. Algorithms 11 and 12 show HATEN2-Tucker-DRI, and HATEN2-PARAFAC-DRI, respectively. HATEN2-DRI has two main ideas: integrating (1) vector products into a matrix product, and (2) products for different factor matrices.

Algorithm 11: HATEN2-Tucker-DRI for computing

$$\mathcal{Y} \leftarrow \mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$$

Input: Tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ and factor matrices $\mathbf{B} \in \mathbb{R}^{J \times Q}$, $\mathbf{C} \in \mathbb{R}^{K \times R}$

Output: Tensor $\mathcal{Y} \in \mathbb{R}^{I \times Q \times R}$

1: $(\mathcal{J}', \mathcal{J}'') \leftarrow \text{IMHP}(\mathcal{X}, \mathbf{B}, \mathbf{C})$;

2: $\mathcal{Y} \leftarrow \text{CrossMerge}(\mathcal{J}', \mathcal{J}'')_{(1)}$;

Algorithm 12: HATEN2-PARAFAC-DRI for computing

$$\mathcal{Y} \leftarrow \mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B})$$

Input: Tensor $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ and factor matrices $\mathbf{B} \in \mathbb{R}^{J \times R}$, $\mathbf{C} \in \mathbb{R}^{K \times R}$

Output: Tensor $\mathcal{Y} \in \mathbb{R}^{I \times R}$

1: $(\mathcal{J}', \mathcal{J}'') \leftarrow \text{IMHP}(\mathcal{X}, \mathbf{B}, \mathbf{C})$;

2: $\mathcal{Y} \leftarrow \text{PairwiseMerge}(\mathcal{J}', \mathcal{J}'')_{(1)}$;

Integrating vector products into a matrix product HATEN2-DRI performs several n -mode vector Hadamard products together in one MAPREDUCE job, instead of multiple jobs, using the n -mode matrix Hadamard product which we define as follows.

Definition 5 (*n -mode matrix Hadamard product*) The n -mode matrix Hadamard product of a tensor

$\mathcal{X} \in \mathbb{R}^{I_1 \times I_2 \times \dots \times I_N}$ with a matrix $\mathbf{U} \in \mathbb{R}^{Q \times I_n}$ is denoted by $\mathcal{X} *_n \mathbf{U}$ and is of size $I_1 \times I_2 \times \dots \times I_N \times Q$. It is defined by

$$(\mathcal{X} *_n \mathbf{U})_{i_1 i_2 \dots i_N q} = (\mathcal{X}^*_{i_1 i_2 \dots i_N} \mathbf{U}_{q, \cdot}^T)_{i_1 i_2 \dots i_N}.$$

where $\mathbf{U}_{q, \cdot}$ is the q th row of \mathbf{U} .

MAPREDUCE algorithm The MAPREDUCE algorithm of n -mode matrix Hadamard product is as follows:

$$\langle \mathcal{X} *_2 \mathbf{B} \rangle$$

- **MAP**: map $\langle i, j, k, \mathcal{X}(i, j, k) \rangle$ on j , and $\langle j, q, \mathbf{B}(j, q) \rangle$ on j such that tuples with the same key are shuffled to the same reducer in the form of <key: j , values: $\{(q, \mathbf{B}(j, q)) \mid \forall q \in \{1, \dots, Q\}\}, \{(i, k, \mathcal{X}(i, j, k)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{:,j,:})\}$ >.
- **REDUCE**: take <key: j , values: $\{(q, \mathbf{B}(j, q)) \mid \forall q \in \{1, \dots, Q\}\}, \{(i, k, \mathcal{X}(i, j, k)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{:,j,:})\}$ > and emit $\langle i, j, k, q, \mathcal{X}(i, j, k)\mathbf{B}(j, q) \rangle$ for each $(i, j, k) \in \text{idx}(\mathcal{X}_{:,j,:})$ and $q \in \{1, \dots, Q\}$.

The previous HATEN2-DRN method performs the n -mode vector Hadamard product $\mathcal{X} *_2 \mathbf{b}_q^T$ for Q times using $\text{nnz}(\mathcal{X}_{:,j,:}) + 1$ of memory space per reducer; however, our new method HATEN2-DRI performs $\mathcal{X} *_2 \mathbf{B}^T$ only once using $\text{nnz}(\mathcal{X}_{:,j,:}) + Q$ of memory space per reducer where Q is used for storing a column of \mathbf{B}^T . HATEN2-DRI decreases the number of jobs significantly without introducing too much overhead since Q is very small (e.g., 10 or 20).

Integrating products for different factor matrices HATEN2-DRI also integrates $\mathcal{X} *_2 \mathbf{B}^T$ and $\text{bin}(\mathcal{X}) *_3 \mathbf{C}^T$ computations into one MAPREDUCE job. This integration is possible since the dependency of the two operations is removed in HATEN2-DRN (and, hence in HATEN2-DRI). Thanks to the integration, the original tensor data \mathcal{X} needs to be read from disks only once (not twice as in previous methods), and thus, we further decrease the running time.

MAPREDUCE algorithm The MAPREDUCE algorithm of the integrating operation, denoted by $\text{IMHP}(\mathcal{X}, \mathbf{B}, \mathbf{C})$, is as follows: $\text{IMHP}(\mathcal{X}, \mathbf{B}, \mathbf{C})$

- **MAP**: map $\langle i, j, k, \mathcal{X}(i, j, k) \rangle$ on j , $\langle j, q, \mathbf{B}(j, q) \rangle$ on j , $\langle i, j, k, \mathcal{X}(i, j, k) \rangle$ on k , and $\langle k, r, \mathbf{C}(k, r) \rangle$ on k such that tuples with the same key are shuffled to the same reducer in the form of <key: j , values: $\{(q, \mathbf{B}(j, q)) \mid \forall q \in \{1, \dots, Q\}\}, \{(i, k, \mathcal{X}(i, j, k)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{:,j,:})\}$ >, and <key: k , values: $\{(r, \mathbf{C}(k, r)) \mid \forall r \in \{1, \dots, R\}\}, \{(i, j, \mathcal{X}(i, j, k)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{:,k,:})\}$ >.
- **REDUCE**: take <key: j , values: $\{(q, \mathbf{B}(j, q)) \mid \forall q \in \{1, \dots, Q\}\}, \{(i, k, \mathcal{X}(i, j, k)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{:,j,:})\}$ > and emit $\langle i, j, k, q, \mathcal{X}(i, j, k)\mathbf{B}(j, q) \rangle$ for each $(i, j, k) \in \text{idx}(\mathcal{X}_{:,j,:})$ and $q \in \{1, \dots, Q\}$, and take <key: k , values: $\{(r, \mathbf{C}(k, r)) \mid \forall r \in \{1, \dots, R\}\}, \{(i, j, \mathcal{X}(i, j, k)) \mid \forall(i, j, k) \in \text{idx}(\mathcal{X}_{:,k,:})\}$ > and emit $\langle i, j, k, r, \mathbf{C}(k, r) \rangle$ for each $(i, j, k) \in \text{idx}(\mathcal{X}_{:,k,:})$ and $r \in \{1, \dots, R\}$.

Finally, we note that although the two decompositions Tucker and PARAFAC are different, our HATEN2-DRI uni-

Table 5 Summary of costs in the steps of all methods for computing $\mathcal{X} \times_2 \mathbf{B} \times_3 \mathbf{C}$ in Tucker decomposition. \mathcal{T} denotes $\mathcal{X} \times_2 \mathbf{B}$

| Method | Step | Intermediate data | Jobs |
|---------------------|--|---|------|
| HATEN2-Tucker-Naive | $\mathcal{X} \times_2 \mathbf{b}_q^T$ | $nnz(\mathcal{X}) + IJK$ | Q |
| | $\mathcal{T} \times_3 \mathbf{c}_r^T$ | $nnz(\mathcal{T}) + IQK$ | R |
| HATEN2-Tucker-DNN | $\mathcal{X} \times_2 \mathbf{b}_q^T$ | $nnz(\mathcal{X}) + J$ | Q |
| | <i>Collapse</i> | $nnz(\mathcal{X})Q$ | 1 |
| | $\mathcal{T} \times_3 \mathbf{c}_r^T$ | $nnz(\mathcal{X})Q + K$ | R |
| | <i>Collapse</i> | $nnz(\mathcal{X})QR$ | 1 |
| HATEN2-Tucker-DRN | $\mathcal{X} \times_2 \mathbf{b}_q^T$ | $nnz(\mathcal{X}) + J$ | Q |
| | $bin(\mathcal{X}) \times_3 \mathbf{c}_r^T$ | $nnz(\mathcal{X}) + K$ | R |
| | <i>CrossMerge</i> | $nnz(\mathcal{X})Q + nnz(\mathcal{X})R$ | 1 |
| HATEN2-Tucker-DRI | <i>IMHP</i> | $2nnz(\mathcal{X}) + JQ + KR$ | 1 |
| | <i>CrossMerge</i> | $nnz(\mathcal{X})Q + nnz(\mathcal{X})R$ | 1 |

Table 6 Summary of costs in the steps of all methods for computing $\mathbf{X}_{(1)}(\mathbf{C} \odot \mathbf{B})$ in PARAFAC decomposition. \mathcal{T}_r denotes $\mathcal{X} \times_2 \mathbf{b}_r^T$

| Method | Step | Intermediate data | Jobs |
|----------------------|--|-------------------------------|------|
| HATEN2-PARAFAC-Naive | $\mathcal{X} \times_2 \mathbf{b}_r^T$ | $nnz(\mathcal{X}) + IJK$ | R |
| | $\mathcal{T}_r \times_3 \mathbf{c}_r^T$ | $nnz(\mathcal{T}_r) + IK$ | R |
| HATEN2-PARAFAC-DNN | $\mathcal{X} \times_2 \mathbf{b}_r^T$ | $nnz(\mathcal{X}) + J$ | R |
| | <i>Collapse</i> | $nnz(\mathcal{X})$ | R |
| | $\mathcal{T}_r \times_3 \mathbf{c}_r^T$ | $nnz(\mathcal{T}_r) + K$ | R |
| | <i>Collapse</i> | $nnz(\mathcal{T}_r)$ | R |
| HATEN2-PARAFAC-DRN | $\mathcal{X} \times_2 \mathbf{b}_r^T$ | $nnz(\mathcal{X}) + J$ | R |
| | $bin(\mathcal{X}) \times_3 \mathbf{c}_r^T$ | $nnz(\mathcal{X}) + K$ | R |
| | <i>PairwiseMerge</i> | $2nnz(\mathcal{X})R$ | 1 |
| HATEN2-PARAFAC-DRI | <i>IMHP</i> | $2nnz(\mathcal{X}) + JR + KR$ | 1 |
| | <i>PairwiseMerge</i> | $2nnz(\mathcal{X})R$ | 1 |

fies them in a general framework of IMHP and merge. As seen in Algorithms 11 and 12, as well as in Fig. 4, HATEN2-Tucker-DRI and HATEN2-PARAFAC-DRI differ only in the merge function (*CrossMerge* for HATEN2-Tucker-DRI and *PairwiseMerge* for HATEN2-PARAFAC-DRI). This general framework allows easier extension of the method for other algorithms, as well as simple maintenance of the code.

3.2.5 Cost of HATEN2

We present the costs of the steps of all HATEN2 methods in terms of the intermediate data size, the number of MAPREDUCE jobs, and the number of the floating point operations in Tables 5 and 6. Based on Tables 5 and 6, we compare the total costs of all the methods in terms of the maximum intermediate data size, and the number of total MAPREDUCE jobs in Tables 3 and 4. We replace $nnz(\mathcal{T} = \mathcal{X} \times_2 \mathbf{B})$ with $nnz(\mathcal{X})Q$ according to the estimation of $nnz(\mathcal{X} \times_2 \mathbf{B})$ in Lemma 4.

Lemma 4 Given a sparse $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ and a fully dense $\mathbf{B} \in \mathbb{R}^{J \times Q}$, the first-order Taylor approximation of the number of nonzeros in $\mathcal{X} \times_2 \mathbf{B}$ is $nnz(\mathcal{X})Q$.

Proof Let $P(\mathcal{X}_{ijk})$ be the probability that $\mathcal{X}_{ijk} \neq 0$. Assuming uniform distribution, $P(\mathcal{X}_{ijk})$ is estimated to be $\frac{nnz(\mathcal{X})}{IJK}$. Since \mathbf{B} is a fully dense matrix, a nonzero element in $\mathcal{X}_{i:k}$ fiber, when multiplied with \mathbf{B} , appears as Q nonzero elements in the result tensor $\mathcal{X} \times_2 \mathbf{B}$. Thus, the probability that there is no element in the (i, k) th fiber of $\mathcal{X} \times_2 \mathbf{B}$ is given by $Q(1 - P(\mathcal{X}_{ijk}))^J = Q(1 - \frac{nnz(\mathcal{X})}{IJK})^J$. Then the estimated number of nonzero elements in $\mathcal{X} \times_2 \mathbf{B}$ is given by $(1 - (1 - \frac{nnz(\mathcal{X})}{IJK})^J) \times IQK$. Applying the first-order Taylor expansion of $(1+x)^n \approx 1+nx$ to the equation $(1 - \frac{nnz(\mathcal{X})}{IJK})^J$, we get

$$\begin{aligned} & \left(1 - \left(1 - \frac{nnz(\mathcal{X})}{IJK} \right)^J \right) \times IQK \\ & \approx \left(1 - \left(1 - J \frac{nnz(\mathcal{X})}{IJK} \right) \right) \times IQK = nnz(\mathcal{X})Q \end{aligned}$$

Note that in Tucker decomposition, HATEN2-Tucker-DRI which contains all the proposed ideas, has the minimum intermediate data size. In PARAFAC, the intermediate data size of HATEN2-PARAFAC-DNN seems smaller than that of

HATEN2-PARAFAC-DRI. However, HATEN2-PARAFAC-DNN has a lower scalability since the matrix \mathcal{J}_r becomes dense, and thus, $\mathcal{J}_r \bar{\times}_3 \mathbf{c}_r^T$ might raise an out of memory error. In contrast, HATEN2-PARAFAC-DRI scales well by exploiting the sparsity of real-world tensors with the idea described in Sects. 3.2.3 and 3.2.4. For both decompositions, HATEN2-DRI has the minimum number of jobs.

3.3 Extensions of HATEN2 for nonnegativity constraints

In this section, we extend HATEN2 for nonnegativity constraints. We first propose HATEN2-PARAFACNN, a distributed nonnegative PARAFAC decomposition whose single-machine version is shown in Algorithm 3. Then, we propose HATEN2-TuckerNN, a distributed nonnegative Tucker decomposition whose single-machine version is shown in Algorithm 4. Similar to the standard PARAFAC and Tucker, the challenges in designing distributed nonnegative factorization algorithms are computing $\mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B})$ and $\mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$, respectively, which incur the intermediate data explosion problem. Using the same ideas presented in Sect. 3, we reduce the size of the intermediate data as well as the number of the total jobs.

3.3.1 HATEN2-PARAFACNN

MapReduce Algorithm for HATEN2-PARAFACNN. Updating each factor of \mathbf{A} , \mathbf{B} , and \mathbf{C} in lines 3, 5, and 7 of Algorithm 3 requires four computational steps:

<Updating factor \mathbf{A} >

- Step 1: $\mathbf{M}_1 \leftarrow \mathbf{X}_{(1)} (\mathbf{C} \odot \mathbf{B})$.
- Step 2: $\mathbf{M}_2 \leftarrow \mathbf{C}^T \mathbf{C} * \mathbf{B}^T \mathbf{B}$.
- Step 3: $\mathbf{M}_3 \leftarrow \frac{1}{\mathbf{AM}_2}$.
- Step 4: $\mathbf{A} \leftarrow \mathbf{A} * \mathbf{M}_1 * \mathbf{M}_3$.

Step 1 uses the same MapReduce algorithm described in Sect. 3.2. In Step 2, the operations $\mathbf{C}^T \mathbf{C}$ and $\mathbf{B}^T \mathbf{B}$ utilize the parallel outer product technique of GigaTensor [18]. Since results of both operations are $R \times R$ matrices and R is small, Hadamard product becomes straightforward. Step 3 computes $\mathbf{A} * \mathbf{M}_2$ where \mathbf{M}_2 is $\mathbf{C}^T \mathbf{C} * \mathbf{B}^T \mathbf{B}$. Using the distributed cache multiplication described in GigaTensor [18], each machine receives \mathbf{M}_2 and calculates each row of the result matrix $\frac{1}{\mathbf{AM}_2}$ with the following map-only job.

- MAP-3: map $\langle i, \mathbf{A}(i, :) \rangle$ on i and calculate the i th row of $\frac{1}{\mathbf{AM}_2}$ with distributively cached \mathbf{M}_2 so that the i th row of $\frac{1}{\mathbf{AM}_2}$ is produced by each mapper.

Step 4 computes the Hadamard product of the three matrices \mathbf{A} , \mathbf{M}_1 , and \mathbf{M}_3 .

- MAP-4: map $\langle i, \mathbf{A}(i, :) \rangle$, $\langle i, \mathbf{M}_1(i, :) \rangle$, and $\langle i, \mathbf{M}_3(i, :) \rangle$ on i such that tuples with the same i are shuffled to the same reducer in the form of $\langle i, \{(\mathbf{A}(i, j), \mathbf{M}_1(i, j), \mathbf{M}_3(i, j) | \forall j)\} \rangle$.
- REDUCE-4: take $\langle i, \{(\mathbf{A}(i, j), \mathbf{M}_1(i, j), \mathbf{M}_3(i, j) | \forall j)\} \rangle$, and emit $\langle i, \mathbf{A}(i, :) * \mathbf{M}_1(i, :) * \mathbf{M}_3(i, :) \rangle$.

3.3.2 HATEN2-TuckerNN

MapReduce Algorithm for HATEN2-TuckerNN. Updating each factor of \mathbf{A} , \mathbf{B} , and \mathbf{C} in lines 3, 5, and 7 of Algorithm 4 requires five computational steps:

<Updating factor \mathbf{A} >

- Step 1: $\mathbf{M}_1 \leftarrow \mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T$.
- Step 2: $\mathbf{M}_2 \leftarrow (\mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T)_{(1)} \mathbf{G}_{(1)}^T$.
- Step 3: $\mathbf{M}_3 \leftarrow (\mathcal{G} \times_2 \mathbf{B}^T \mathbf{B} \times_3 \mathbf{C}^T \mathbf{C})_{(1)} \mathbf{G}_{(1)}^T$.
- Step 4: $\mathbf{M}_4 \leftarrow \frac{1}{\mathbf{AM}_3}$.
- Step 5: $\mathbf{A} \leftarrow \mathbf{A} * \mathbf{M}_2 * \mathbf{M}_4$.

Step 1 uses the same MapReduce algorithm as described in Sect. 3.2. Step 2 multiplies $(\mathcal{X} \times_2 \mathbf{B}^T \times_3 \mathbf{C}^T)_{(1)}$ and $\mathbf{G}_{(1)}^T$ which requires a similar MapReduce job of Step 3 in HATEN2-PARAFACNN. In Step 3, the operations $\mathbf{B}^T \mathbf{B}$ and $\mathbf{C}^T \mathbf{C}$ utilize the parallel outer product technique of GigaTensor [18] as explained in Step 2 of HATEN2-PARAFACNN. Since the sizes of the core tensor $\mathbf{B}^T \mathbf{B}$, and $\mathbf{C}^T \mathbf{C}$ are small, we perform $(\mathcal{G} \times_2 \mathbf{B}^T \mathbf{B} \times_3 \mathbf{C}^T \mathbf{C})_{(1)} \mathbf{G}_{(1)}^T$ in a local machine. Steps 4 and 5 are the same as those in Steps 3 and 4 of HATEN2-PARAFACNN, respectively. Updating the core tensor \mathcal{G} in line 9 of Algorithm 4 requires similar computation steps.

4 Experiment

In this section, we present experimental results to answer the following questions:

- Q1** What is the performance of HATEN2-DRI compared with other methods?
- Q2** How well does HATEN2-DRI scale up with various factors (nonzeros, dimensionality, density, core tensor size, order, and machines)?

After describing the experimental settings in Sect. 4.1, we present the scalability results in Sect. 4.2 to answer **Q1** and **Q2**.

4.1 Experimental settings

We compare our final method HATEN2-DRI with other methods (HATEN2-Naive, HATEN2-DNN, HATEN2-DRN) as well

as the Tensor Toolbox [21], the state-of-the-art tensor computation package for a single machine.

4.1.1 Machines

HATEN2 is run on a HADOOP cluster with 40 machines where each machine has a quad-core Intel Xeon E3 1230v3 3.3Ghz CPU, 32 GB RAM, and 12 Terabytes disk. The Tensor Toolbox is run on a machine from the HADOOP cluster.

4.1.2 Dataset

The tensor dataset used in our experiments is summarized in Table 7, with the following details.

- Freebase-sampled: sampled RDF triples (subject entity, object entity, relation) where the entries are related with music, book, tv shows, film, people, and sport from Freebase [19].
- Phonecall: real-world phone call history data (who-calls-whom) containing (sender, receiver, date) triples (e.g., ‘1234,’ ‘5678,’ ‘06-DEC-07’) in 2007-12-01 to 2008-1-31.
- NELL: real-world knowledge base data containing (noun phrase 1, noun phrase 2, context) triples (e.g., ‘George Harrison,’ ‘guitars,’ ‘plays’) from the ‘Read the Web’ project [1]. NELL-2 data are the filtered data from NELL by removing entries whose values are below a threshold.
- DARPA1998: 1998 DARPA intrusion detection evaluation dataset from MIT Lincoln Laboratory [20]. We translate the packet data into a 3-way tensor which is composed of (source IP, destination IP, time) triples. For example, a triple (‘202.77.162.213,’ ‘172.16.114.50,’ ‘1998-06-19 08:49:21,’ ‘1’) corresponds to a packet sent from 202.77.162.213 to 172.16.114.50 at 08:49:21, June 11, 1998.
- Random: synthetic random tensor of size $I \times I \times I$. The size I varies from 10^3 to 10^8 , the number of nonzeros varies from 10^4 to 10^{10} , and the density varies from $10^{-15} \sim 10^{-5}$.

4.2 Scalability

To answer the questions Q1 and Q2, we compare the machine and the data scalabilities of HATEN2-DRI with those of other methods. Since the Tensor Toolbox does not provide nonnegative tucker decomposition, we omit scalability experiments of the Tensor Toolbox.

Table 7 Summary of the tensor data used

| Data | I | J | K | Nonzeros | Density | Format | Source |
|------------------|-----------|-----------|-----------|-----------|-------------------------|-----------------------------------|-----------------------|
| Freebase-sampled | 38 M | 38 M | 532 | 139 M | 1.733×10^{-10} | {entity, entity, relation} | Freebase [19] |
| Phonecall | 30 M | 30 M | 62 | 184 M | 3.297×10^{-9} | {sender, receiver, time} | |
| NELL | 26 M | 26 M | 48 M | 144 M | 4.387×10^{-15} | {subject, object, predicate} | ‘Read the Web’ [1] |
| NELL-2 | 15 K | 15 K | 29 K | 77 M | 1.262×10^{-5} | {subject, object, predicate} | ‘Read the Web’ [1] |
| DARPA1998 | 22 K | 22 K | 23 M | 28 M | 2.515×10^{-9} | {source IP, destination IP, time} | MIT Lincoln Lab. [20] |
| Random | 1 K~100 M | 1 K~100 M | 1 K~100 M | 10 K~10 B | $10^{-15} \sim 10^{-5}$ | | |

B billion, *M* million, *K* thousand

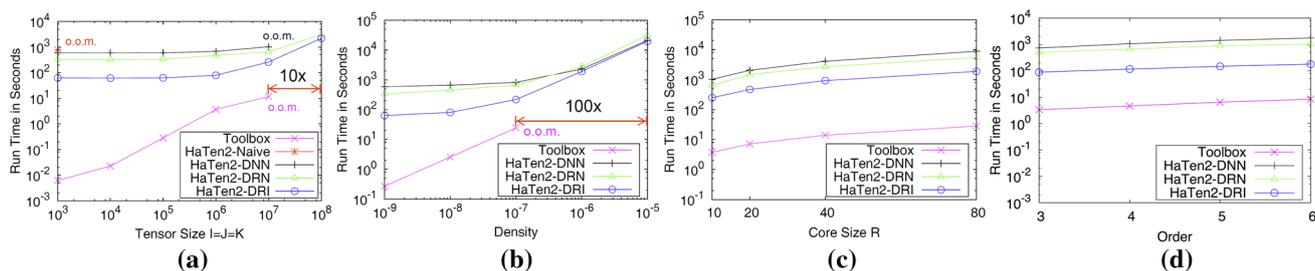


Fig. 7 Data scalability of our proposed HATEN2-DRI compared to other methods, for PARAFAC decomposition. The datasets are explained in detail at Sect. 4.1. o.o.m.: out of memory. Note that our

4.2.1 Data scalability

Data scalability is measured for the following four aspects: number of nonzeros and dimensionality, density, core tensor size, and order. We change the input tensor in terms of each aspect one by one, while fixing other aspects, and measure the running time using all the 40 machines in the cluster. Since the HATEN2-Naive method cannot process even a 10^4 scale tensor (Figs. 1a, 7a), HATEN2-Naive is omitted from the density and core scalability experiments (Figs. 1b, c, 7, 8 and 9b, c).

Nonzeros and dimensionality We increase the dimensionality $I = J = K$ of modes from 10^3 to 10^8 . The number of nonzeros is set to dimensionality $\times 10$. For Tucker decomposition, the size $P \times Q \times R$ of the core tensor is fixed to $10 \times 10 \times 10$. For PARAFAC decomposition, the rank R is set to 10. As shown in Figs. 1a, 7, 8 and 9a, our best method HATEN2-DRI shows the best result: HATEN2-DRI analyzes 10^8 scale tensor the most quickly. HATEN2-Naive and HATEN2-DNN failed for tensors with size beyond 10^3 and 10^7 , respectively. Although HATEN2-DRN also analyzes 10^8 scale tensor, the running time is 1.7 times slower than that of HATEN2-DRI.

Density We increase the density (= number of nonzeros / number of all possible elements) of input tensor from 10^{-9} to 10^{-5} ; accordingly, the number of nonzeros becomes 1 billion to 1 trillion, and they take 20 MB to 196 GB disk space. The dimensionality of each mode is set to 10^5 ($I = J = K$). For Tucker decomposition, the size $P \times Q \times R$ of the core tensor is fixed to $10 \times 10 \times 10$. For PARAFAC decomposition, the rank R is set to 10. As shown in Figs. 1b, 7, 8 and 9b, our method HATEN2-DRI analyzes up to $1000\times$ denser tensors than existing method for nonnegative PARAFAC decomposition and the running time is the fastest compared with other variants of HATEN2.

Core tensor size We increase the core size of a random tensor of size $10^6 \times 10^6 \times 10^6$ with 10^7 nonzeros, and measure the running time. For Tucker decomposition, the core tensor size increases from $10 \times 10 \times 10$ to $40 \times 40 \times 40$ or $80 \times 80 \times 80$; for PARAFAC, the rank R increases from 10

best method HATEN2-DRI decomposes $10 \sim 100\times$ larger data than the Tensor Toolbox. **a** Nonzeros and dimensionality, **b** Density, **c** Core, **d** Order

to 40 or 80. For normal PARAFAC and Tucker decompositions, as shown in Figs. 1c and 7c, HATEN2-DRI scales well, providing the best performance for all the core sizes. When the core size is 80, HATEN2-DRI outperforms the second best method (HATEN2-DRN) by 2.25 times. For nonnegative PARAFAC and Tucker decompositions, as shown in Figs. 8c and 9c, HATEN2-DRI scales well up to core size 40. Note that in Fig. 9c, HATEN2-DRI decomposes a tensor with a $4\times$ larger core tensor compared to the Tensor Toolbox.

Order We increase the order (=number of modes) of a random tensor of size $10^6 \times 10^6 \times 10^6$ with 10^7 nonzeros, and measure the running time. For normal PARAFAC and Tucker decompositions, as shown in Figs. 1d and 7d, HATEN2-DRI scales well, providing the best performance among variants of HATEN2 for all the orders. Similar scalability is observed for nonnegative PARAFAC and Tucker decompositions, as shown in Figs. 8d and 9d, respectively.

4.2.2 Machine scalability

To measure the machine scalability, we increase the number of machines from 10 to 40, and report T_{10}/T_M where T_M is the running time with M machines. We use the NELL tensor data of size $26 \text{ M} \times 26 \text{ M} \times 48 \text{ M}$ containing 144 M nonzeros. For Tucker and nonnegative Tucker decomposition, the core tensor size is set to $10 \times 10 \times 10$. For PARAFAC and nonnegative PARAFAC decomposition, the rank size is set to 10. As shown in Fig. 10, our best method HATEN2-DRI scales near linearly in the beginning, while the performance flattens as the number of machines grows due to overheads required in distributed systems (e.g., synchronization time, JVM loading time). Note that the machine scalability of Tucker is better than that of PARAFAC. Since Tucker is more complex and involves a larger amount of computations compared with PARAFAC, increasing the number of machines is more effective in Tucker. Due to many sub-operations performed in a local machine, the machine scalabilities of nonnegativity-constrained Tucker and PARAFAC decompositions are smaller than those of the unconstrained versions.

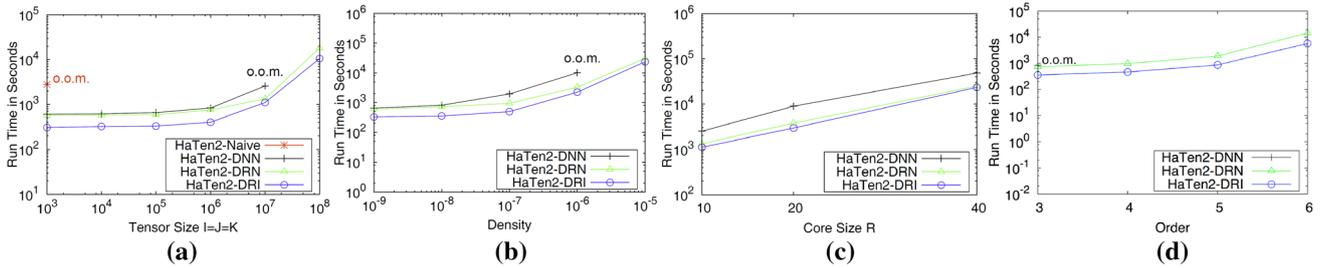


Fig. 8 Data scalability of our proposed HATEN2-DRI compared to other methods, for nonnegative Tucker decomposition. The datasets are explained in detail at Sect. 4.1. o.o.m.: out of memory. Among the vari-

ants of HATEN2, HATEN2-DRI is the fastest, and analyzes $10\times$ denser data than HATEN2-DNN does. **a** Nonzeros and dimensionality, **b** Density, **c** Core, **d** Order

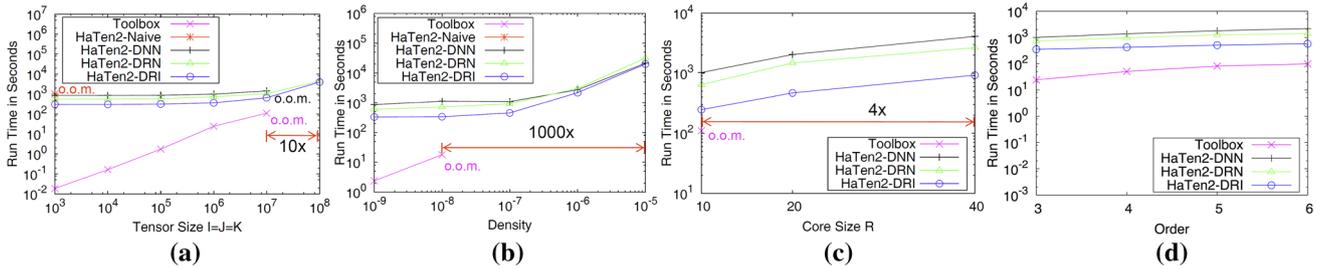


Fig. 9 Data scalability of our proposed HATEN2-DRI compared to other methods, for nonnegative PARAFAC decomposition. The datasets are explained in detail at Sect. 4.1. o.o.m.: out of memory. Note that HATEN2-DRI decomposes $10 \sim 1000\times$ larger data than the Tensor Toolbox and HATEN2 decomposes a tensor with a $4\times$ larger core ten-

sor compared to the Tensor Toolbox. Among the variants of HATEN2, HATEN2-DRI is the fastest, and analyzes $10\times$ larger data than HATEN2-DNN does. **a** Nonzeros and dimensionality, **b** Density, **c** Core, **d** Order

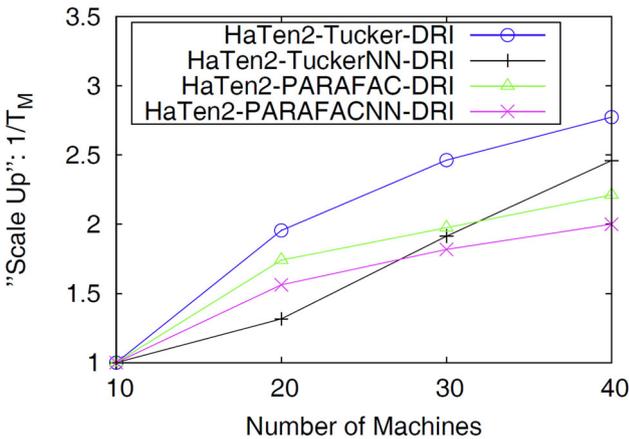


Fig. 10 Machine scalability of HATEN2-PARAFAC-DRI, HATEN2-PARAFACNN-DRI, HATEN2-Tucker-DRI, and HATEN2-TuckerNN-DRI with regard to the ‘Scale Up’ factor $\frac{T_{10}}{T_M}$, where T_M is the running time with M machines. Note that in all cases, HATEN2-DRI scales near linearly in the beginning, while the performance flattens as more machines are added due to overheads in distributed systems

5 Discovery

In this section, we present discovery results to answer the following questions:

- Q1** What are the discoveries on real-world tensors?
- Q2** What are the differences between PARAFAC and Tucker decompositions on real-world tensors?

We apply HATEN2 to four large-scale real-world tensors: Freebase-sampled, DARPA1998, Phnecall, and NELL-2.

5.1 Freebase-sampled

Freebase [19] is a knowledge base dataset in the RDF (Resource Description Framework) format, composed of (subject entity, object entity, relation) triples. Below, we explain construction of a tensor from Freebase and interesting concepts discovered by HATEN2.

Building Freebase-sampled tensor We build a Freebase-sampled tensor from the full Freebase dataset [19]. First, we extract relations about several important topics including ‘Music,’ ‘Books,’ ‘People,’ ‘Film,’ ‘TV,’ and ‘Sports.’ Then, we remove the triples containing literal entities (e.g. (John, ‘John,’ name)) since they represent definitions which do not help reveal latent concepts.

Concept discovery We find several latent concepts in the Freebase-sampled tensor by applying HATEN2-Tucker and HATEN2-PARAFAC on it. We choose the top- k highest val-

Table 8 Concept discovery result from HATEN2-PARAFAC on the Freebase-sampled dataset

| Concepts | Subject entity | Object entity | Relation |
|-----------------------------|--------------------------|---|------------------------------------|
| Concept1: 'Pop/Rock Albums' | Pop Music | The Day Hell Broke Loose at Sicard Hollow | ns:music.album_release_type.album |
| | Alternative rock | Enigma variations | ns:music.artist.track |
| | Rock music | A bunch of stuff | ns:music.genre.album |
| Concept2: 'Expert' | | The fifteenth porn cut | ns:music.artist.album |
| | Actor | Lia scott price | |
| | Film Producer | Eric idle | ns:people.profession |
| | Comedian | Barbra Streisand | People_with_this_profession |
| Concept3: 'Classic' | Singer | Madonna | |
| | Johann Sebastian Bach | Concerto for oboe & violin in D minor, BWV 1060: II. Adagio | ns:music.artist.album |
| | Pyotr Ilyich Tchaikovsky | Prelude No. 10 in E minor, BWV 855 | ns:music.artist.track_contribution |
| | John Eliot Gardiner | Choral: 'Ach mein herzliebes Jesulein' | ns:music.genre.album |
| | Heinrich Schiff | Cello Suite No.4 Es-dur BWV 1010: III. Courante | ns:music.album_release_type.album |

Table 9 Discovered factors from HATEN2-Tucker on the Freebase-sampled dataset

| | Subject S1: instruments | Subject S2: jobs | Subject S3: marriage |
|----------------|---|--|--|
| Subject entity | Piano Guitar Electronic keyboard Bass | Actor Film score composer Singer Screenwriter | Marriage |
| | Object O1: recording contributors | Object O2: persons | Object O3: spouse |
| Object entity | Pedro Rousseau Yann Tiersen Ari Hest Krischan Frehse | O. Z. Livaneli Eric Idle Jay Chou Haylar Garcia | Helene Belmar Julius & William Safire Beverly McKittrick & Jackie Gleason Kathleen Garman & Jacob Epstein Tyrone Willingham & Kim |
| | Relation R1: professionalism | Relation R2: marriage | Relation R3: book |
| Relation | ns:people.profession .people_with_this_profession | ns:people.marriage_union_type .unions_of_this_type | ns:book.newspaper_issue.publication_date ns:book.poetic_verse_form.poems_of_this_form ns:book.magazine.genre ns:book.book_edition.interior_illustrations_by |

ued elements from each column of each factor. Table 8 shows the results by HATEN2-PARAFAC with rank 10. There are several concepts (e.g., 'Pop/Rock Music Albums,' 'Expert,' and 'Classic') each of which contains groups of subjects, objects, and relations. Note that each subject group is tightly coupled only with the corresponding object and relation groups, due to the diagonal core tensor of PARAFAC. On the other hand, Tucker decomposition gives more diverse concepts determined by cross combinations of various groups. Table 9 shows factors by HATEN2-Tucker with the core size $10 \times 10 \times 10$. We find several groups for each mode: e.g., for the 'subject' mode, we find the groups of 'Instruments,'

'Jobs,' and 'Marriage.' Table 10 shows concepts each of which combines groups from the subject, the object, and the relation factors. The first concept 'Musician' consists of the subject group S1 ('Instruments'), the object group O1 ('Recording Contributors'), and the relation group R1 ('Professionalism'); the second concept 'Expert' consists of the subject group S2 ('Jobs'), the object group O2 ('Persons'), and the relation group R1 ('Recording Contributors'). Note that the object group R1 appears in both of the concepts, exemplifying the Tucker's ability of finding concepts from various, possibly overlapping groups. The last concept 'Married Couple' consists of the subject group S3 ('Marriage'),

Table 10 Concept discovery result from HATEN2-Tucker on the Freebase-sampled dataset

| Concepts | Subject entity | Object entity | Relation |
|---------------------------------------|---------------------|---------------------------------------|--------------------------------|
| Concept1:(S1,O1,R1) ‘Musician’ | Piano | Pedro Rousseau | |
| | Guitar | Yann Tersen | ‘ns:people.profession |
| | Electronic keyboard | Ari Hest | .people_with_this_profession’ |
| | Bass | Krischan Frehse | |
| Concept2: (S2,O2,R1) ‘Expert’ | Actor | O. Z. Livaneli | |
| | Film score composer | Eric Idle | ‘ns:people.profession |
| | Singer | Jay Chou | .people_with_this_profession’ |
| | Screenwriter | Haylar Garcia | |
| Concept3: (S3,O3,R2) ‘Married Couple’ | | Helene Belmar Julius & William Safire | |
| | Marriage | Beverly McKittrick & Jackie Gleason | ‘ns:people.marriage_union_type |
| | | Kathleen Garman & Jacob Epstein | .unions_of_this_type’ |
| | | Tyrone Willingham & Kim | |

the object group O3 (‘Spouse’), and the relation group R2 (‘Marriage’).

5.2 DARPA1998

DARPA1998 is an intrusion detection evaluation dataset provided by MIT Lincoln Laboratory [20]. They provide packet dump files containing network traffic logs during seven weeks and a list of network-based attacks. The list of attack information contains descriptions of attacks such as source IPs, destination IPs, attack types, and times. Table 11 shows a summary of the attack information. More detailed descriptions are in [20].

Building network traffic tensor To analyze network traffic logs provided in the DARPA1998 dataset, we convert network packet dump data into a 3-way tensor. First, we extract source and destination IP addresses, and a timestamp for each packet. Next, we map the IP addresses and the timestamps onto natural numbers. Since a timestamp has a continuous value, we discretize it into bins of length 0.001 second. Last, we build a 3-way tensor with (‘Source IP,’ ‘Destination IP,’ ‘Time’) triples where each element in the tensor represents the number of packets for the corresponding triple.

Network traffic pattern discovery By applying HATEN2 on the tensor constructed above, we discover several interesting patterns summarized in Fig. 11. Table 12 shows discovery results by HATEN2-PARAFAC with rank 10. We find several patterns (e.g., ‘neptune attack,’ ‘Heavy interaction,’ and ‘Normal traffic’) from factor groups that contain source IP, destination IP, and time. For each group, we present IP addresses and time that have extraordinarily high scores. In the ‘neptune attack’ and ‘Heavy interaction’ patterns, scores are concentrated on certain IP addresses and times since massive packets are exchanged between several machines at a

Table 11 Descriptions of attacks that occur in the DARPA1998 network traffic logs

| | Description |
|-------------|--|
| Attacker IP | 135.13.216.191 |
| | 10.20.30.40 |
| | 230.1.10.20 |
| | 194.7.248.53 |
| | ... |
| Victim IP | 172.16.112.50 (pascal) |
| | 172.16.113.50 (zeno) |
| | 172.16.114.50 (marx) |
| Duration | 1998-06-01–1998-07-17 |
| Attack type | Denial of service: back, land, neptune, pod, smurf, syslog, teardrop |
| | Sweeping: ip sweep, port sweep |
| | Buffer overflow: eject, ffb, format, imap |
| | Guessing password: dict, guest |
| | etc: ftp-write, loadmodule, nmap, perlmagic, phf ... |

certain point. On the other hand, in the ‘Normal traffic’ pattern, scores are spread over all IP addresses and times, and the score gaps between entries are small. We apply HATEN2-Tucker to the same tensor; we discover patterns for each factor group and find network traffic patterns by combining the factor groups. Table 13 shows several factors discovered by HATEN2-Tucker with the core size $10 \times 10 \times 10$. For the ‘source IP’ mode, we find ‘neptune attacker,’ ‘port sweep attacker’ and ‘Various attacker’ groups. The ‘Various attacker’ group performs various types of attacks several times. Similarly, we find victim groups and attack time groups from the ‘Destination IP’ mode and the ‘Time’ mode, respectively. Since packets are exchanged between attackers

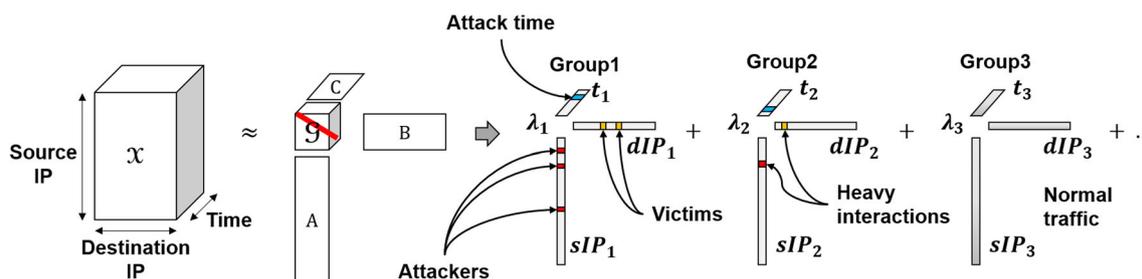


Fig. 11 Network traffic pattern discovery result by HaTen2 on DARPA1998, an intrusion detection dataset. Group1 shows an attack pattern where several attackers perform malicious attacks like denial of

service to the victims at a certain point. Group2 shows an interaction pattern where two machines intensively communicate with each other at a certain point. Group3 shows a normal traffic pattern

and victims, D3 group consists of attackers though it stands for the destination IP address. Table 14 shows the discovered patterns that are combinations of source IP, destination IP, and time factor groups. The first group ‘Heavy interaction’ consists of the source IP group S1 (‘neptune attacker’), the destination IP group D1 (‘Victim’), and the time group T1 (‘Specific time’); The second group ‘neptune attack’ consists of the source IP group S1 (‘neptune attacker’), the destination IP group D1 (‘Victim’), and the time group T2 (‘Specific time’). The third group ‘Various attack’ consists of the source IP group S3 (‘Various attacker’), the destination IP group D2 (‘Victim’), and the time group T3 (‘Various time’). The third group shows various attack patterns performed by several attackers at several occasions. Note that the source IP group S1 and the destination IP group D1 are shared by Pattern1 and Pattern2. Since Tucker provides various concepts by cross combinations of factor groups, some factor groups appear in multiple concepts. We also apply the nonnegativity-constrained Tucker and PARAFAC using our HATEN2-TuckerNN and HATEN2-PARAFACNN. The overall result is similar to that of the unconstrained versions of Tucker and PARAFAC. However, the entries that have negatively high scores in the unconstrained versions are scored positively in the nonnegative tensor decompositions. Since all entries have values larger than zero in nonnegative tensor decomposition, the importance between entities is easily compared. Comparing to the ground truth, we successfully detect attackers and victims in the network traffic logs. For example, we find the following attacker groups: ‘neptune’ attackers, ‘port sweep’ attackers, and attackers who perform various types of attacks. Note that we detect the exact attack time for the ‘neptune’ attack, one of the denial of service attack where attackers send lots of packets to victims intensively at a certain time.

5.3 Phonenumber

Phonenumber is a real-world phone call history data from an anonymous provider, containing the information of senders,

receivers, dates, times, and durations. Below, we explain the construction of a tensor from the Phonenumber data and interesting concepts discovered by HATEN2.

Building Phonenumber tensor To build a 3-way tensor, we extract sender, receiver, and date entries from the Phonenumber dataset, and map them to natural numbers. Then, we build a 3-way tensor with (‘sender,’ ‘receiver,’ ‘date’) triples where each element in the tensor represents the number of calls for the corresponding triple.

Phonenumber pattern discovery Table 15 shows discovered factor groups after applying HaTen2-Tucker on the Phonenumber dataset. We discovered three groups each of which is a combination of factor groups. Figure 12 shows call patterns of those groups, and Table 16 shows in more details how they are composed of. In Table 15, for the sender mode, we find ‘Telemarketer,’ ‘Senders in a closed group’ and ‘Normal’ groups. Corresponding to the ‘Telemarketer’ group, we find the ‘Victim’ group in the receiver mode. The members of the ‘Telemarketer’ group call over all the other people, but never receive calls from anyone. Figure 13 shows the telephone traffic pattern of a telemarketer with ID 19893602 and a victim with ID 3517446. Note that the telemarketer calls 218,725 people 46 times on average, and 3,367 times at maximum. The amount of received calls is zero, because they never receive calls from anyone. On the other hand, the members of the ‘Victim’ group receives many calls from telemarketers, while receiving few calls from normal people. For example, in Fig. 13b, a receiver with ID 3517446 receives 1–10 calls from normal people, but more than 3,000 calls from a telemarketer. These ‘Telemarketer’ and ‘Victim’ groups form the ‘Telemarketing’ group as shown in Table 16. Another group called the ‘Closed group’ consists of two sub-groups: ‘Senders in a closed group’ for the sender mode and ‘Receivers in a closed group’ for the receiver mode. In this group, members of the ‘Senders in a closed group’ send a large amount of traffics to the members of the ‘Receivers in a closed group.’ The last group we discover is the ‘Normal’ group. In the ‘Normal’ group, each member interacts with a small number of people in the group.

Table 12 Network traffic pattern discovery result by HATEN2-PARAFAC on the DARPA1998 dataset

| Traffic pattern | Source IP | Destination IP | Time | Description |
|--------------------------------|----------------|----------------|------------------------|---|
| Pattern1: 'Neptune attack' | 10.20.30.40 | 172.16.112.50 | 1998-07-02, 10:16 A.M. | Neptune: Syn flood denial of service on one or more ports |
| Pattern2: 'Heavy interactions' | 135.13.216.191 | 172.16.112.50 | 1998-07-01, 09:46 A.M. | There were heavy interactions between 2 machines through telnet |
| Pattern3: 'Normal traffic' | 172.16.112.194 | 194.27.251.21 | 1998-06-15, 03:27 P.M. | Normal traffic logs |
| | 172.16.112.50 | 194.7.248.153 | 1998-06-30, 11:57 A.M. | |
| | 172.16.114.148 | 197.218.177.69 | 1998-07-13, 04:56 P.M. | |

Table 13 Discovered factors from HATEN2-Tucker on the DARPA1998 dataset

| | Source IP S1: 'neptune' attacker | Source IP S2: 'port sweep' attackers | Source IP S3: various attackers |
|----------------|--|--|---|
| Source IP | 135.13.216.191 10.20.30.40 230.1.10.20 | 194.7.248.153 194.27.251.21 | 135.8.60.182 197.182.91.233 197.218.177.69 |
| | Destination IP D1: victim | Destination IP D2: victims | Destination IP D3: attackers |
| Destination IP | 172.16.112.50 | 194.7.248.153 172.16.113.50 172.16.112.50 | 172.16.114.148 194.27.251.21 197.218.177.69 |
| | Time T1: specific time | Time T2: specific time | Time T3: various times |
| Time | 1998-07-01, 09:46 A.M. | 1998-07-09, 12:10 P.M. 1998-07-09, 12:22 P.M. 1998-07-09, 12:22 P.M. | 1998-06-11, 2:05 P.M. 1998-06-12, 11:56 A.M. 1998-07-08, 11:16 P.M. |

Table 14 Network traffic pattern discovery result by HATEN2-Tucker on the DARPA1998 dataset

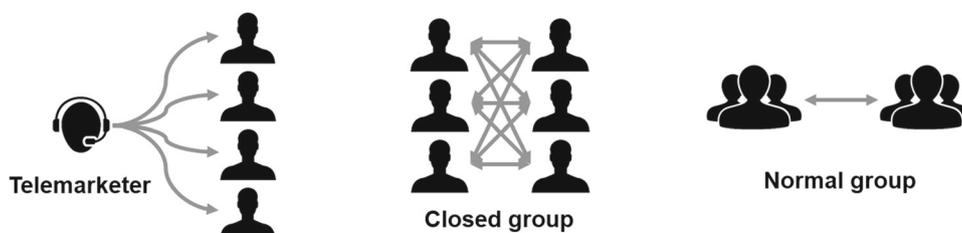
| Traffic pattern | Source IP | Destination IP | Time | Description |
|---|----------------|----------------|------------------------|--|
| Pattern1:(S1, D1, T1) 'heavy interaction' | 135.13.216.191 | 172.16.112.50 | 1998-07-01, 09:46 A.M. | There were heavy interactions between 2 machines through telnet |
| | 10.20.30.40 | | | |
| | 230.1.10.20 | | | |
| Pattern2:(S1, D1, T2) 'neptune attack' | 135.13.216.191 | 172.16.112.50 | 1998-07-09, 12:10 P.M. | Neptune: Syn flood denial of service on one or more ports |
| | 10.20.30.40 | | 1998-07-09, 12:22 P.M. | |
| | 230.1.10.20 | | 1998-07-09, 12:23 P.M. | |
| Pattern3:(S3, D2, T3) 'various attack' | 135.8.60.182 | 194.7.248.153 | 1998-06-11, 2:05 P.M. | Several attackers perform various types of attacks several times |
| | 197.182.91.233 | 172.16.113.50 | 1998-06-12, 11:56 A.M. | |
| | 197.218.177.69 | 172.16.112.50 | 1998-07-08, 11:16 P.M. | |

Lastly, we examine the factors for the date mode. Figure 14 shows patterns of the four date-mode factors described in Table 15. For the date mode, we find 'Normal,' 'Spike,' 'Periodic,' and 'Christmas' groups. Factor1 ('Normal' factor) shows the normal pattern having similar scores over most days, while having high scores before special events

such as Christmas and the New Year's Day. This result makes sense because people usually give words of blessing to their friends right before such special days. Factor2 ('Spike' factor) has an abnormal peak point at 2007-12-07. Factor3 ('Periodic' factor) shows a periodic pattern, and Factor4 ('Christmas' factor) reaches its positive peak right before

Table 15 Discovered factor groups by HATEN2-Tucker on the Phonecall dataset

| | Sender S1: telemarketer | Sender S2: senders in a closed group | Sender S3: normal | |
|-------------|-------------------------|--|-------------------|--------------------|
| Sender_id | 19893602 | 5188590 | 7145415 | |
| | 19922918 | 5188591 | 8349492 | |
| | 6657916 | 5188592 | 1944847 | |
| | Receiver R1: victim | Receiver R2: receivers in a closed group | | |
| Receiver_id | 3517446 | 5188590 | | |
| | 3517450 | 5188591 | | |
| | 4605892 | 5188592 | | |
| | Date D1: normal | Date D2: spike | Date D3: periodic | Date D4: christmas |
| Date | 21-Dec-07 | 07-Dec-07 | 07-Dec-07 | 21-Dec-07 |
| | 03-Jan-08 | | 09-Dec-07 | 22-Dec-07 |
| | 04-Jan-08 | | 27-Jan-08 | 23-Dec-07 |

**Fig. 12** Discovered sender and receiver groups by HATEN2 on the Phonecall dataset. We find telemarketers who call many people a lot, but never receive calls. In the closed group, a large amount of traffics are

internally concentrated. There is also a normal group where a person interacts with a small number of people

Table 16 Phone call pattern discovery result by HATEN2-Tucker on the Phonecall dataset

| Phone call Pattern | Sender | Receiver | Date |
|--|----------|----------|-----------|
| Pattern1:(S1, R1, D1) 'telemarketing' | 19893602 | 3517446 | 21-Dec-07 |
| | 19922918 | 3517450 | 03-Jan-08 |
| | 6657916 | 4605892 | 04-Jan-08 |
| Pattern2:(S2, R2, D3) 'closed group' | 5188590 | 5188590 | 07-Dec-07 |
| | 5188591 | 5188591 | 09-Dec-07 |
| | 5188592 | 5188592 | 27-Jan-08 |
| Pattern3:(S3, R3, D1) 'normal' | 7145415 | 3517446 | 21-Dec-07 |
| | 8349492 | 3517450 | 03-Jan-08 |
| | 1944847 | 4605892 | 04-Jan-08 |

Christmas and negative peak on 2007-12-07. The 'Normal' factor is contained in the 'Normal' and 'Telemarketing' pattern in Table 16, since telemarketers call many normal people. The 'Periodic' factor is contained in the 'Closed group' pattern since the members in the group interacts with each other periodically.

5.4 NELL-2

NELL is a knowledge base dataset containing ('Noun Phrase 1,' 'Noun Phrase 2,' 'Context') triples from the 'Read the Web' project [1]. We filter the NELL data by removing entries whose values are below a threshold; the result is a tensor named NELL-2 whose size is $14545 \times 14545 \times 28818$ with 76 millions of nonzeros.

Concept discovery We discover latent concept groups of NELL-2 by applying HATEN2-PARAFAC with rank 20 and HATEN2-Tucker with the core tensor size $20 \times 20 \times 20$. Table 17 shows the concept discovery results from HATEN2-PARAFAC. We discovered several concepts: e.g., 'Health Care System,' 'File Transfer,' 'Internet Service,' and 'Shopping.' In PARAFAC decomposition, because the core tensor is diagonal, each 'Noun Phrase 1' group is combined only with a 'Noun Phrase 2' group and a 'Context' group. On the other hand, Tucker decomposition provides more diverse concepts compared with PARAFAC decomposition: e.g., a 'Noun Phrase 2' group may be combined with several 'Noun Phrase 1' groups and 'Context' groups. Table 18 shows the groups in factors from Tucker decomposition: e.g., 'Health,' 'Credit,' 'Network,' 'Algorithm,' 'Project,' and 'Information'

Fig. 13 Telephone traffics of one of telemarketers with ID 19893602 and one of victims with ID 3517446. The telemarketer calls over all people, but never receives calls from any other. The victim receives lots of calls from the telemarketer while receiving a few calls from normal people

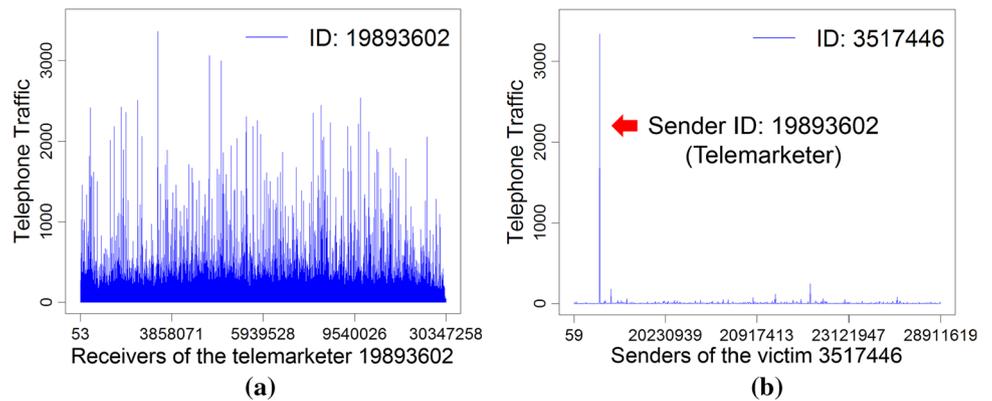
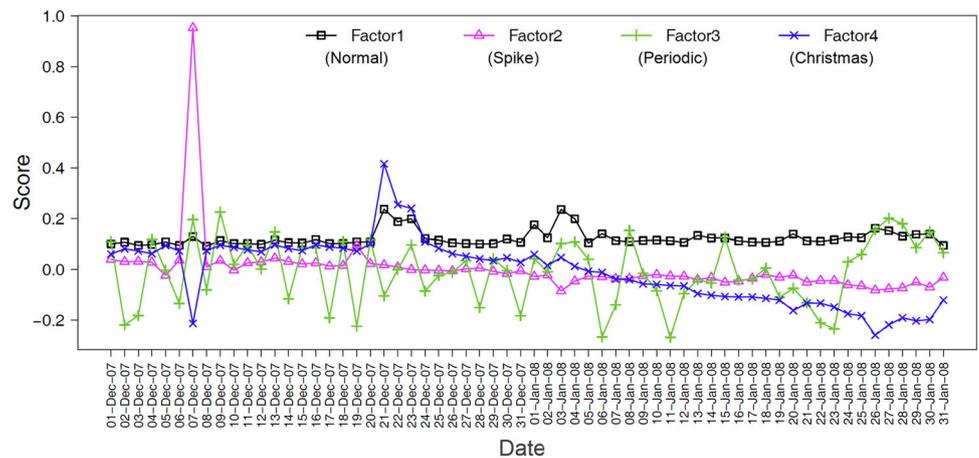


Fig. 14 Discovered patterns for the date factors by HATEN2 on the Phonecall dataset. Factor1 shows a normal pattern that has high scores right before Christmas and the New Year’s Day, and remains constant for the other days; Factor2 has an abnormal peak point on 2007-12-07; Factor3 shows a periodic pattern; Factor4 reaches its positive peak right before Christmas and negative peak on 2007-12-07



in the ‘Noun Phrase 1’ mode. Table 19 shows the discovered concepts each of which combines the groups from the ‘Noun Phrase 1,’ the ‘Noun Phrase 2,’ and the ‘Context’ factors. The first concept represents ‘Health Care System’ which contains the ‘Noun Phrase 1’ group S1 (‘Health’), the ‘Noun Phrase 2’ group O2 (‘Service’), and the ‘Context’ group C1 (‘Care’). Note that a group of a factor appears in several concept groups in Tucker decomposition. For example, the ‘Noun Phrase 2’ group O2 appears in the first, the second, and the third concepts; the ‘Context’ group C6 appears in both the second and the third concepts.

6 Related work

6.1 CP/PARAFAC

Acar et al. [22] use the PARAFAC decomposition in order to detect epilepsy in brain measurements. In [2], Kolda and Bader extend the popular HITS algorithm for ranking web-pages, by incorporating anchor text information to the hyperlinks, and using PARAFAC in order to derive hubs and authorities from the data. PARAFAC has also been used in anomaly detection; [3] and [23] detect network anomalies

Table 17 Concept discovery result using HATEN2-PARAFAC on the NELL-2 dataset

| Concepts | Noun Phrase1 | Noun Phrase2 | Context |
|----------------------|--------------|---------------|-----------------------------|
| Concept1: | health | providers | ‘np1’ ‘care’ ‘np2’ |
| ‘Health Care System’ | child | systems | ‘np1’ ‘insurance’ ‘np2’ |
| | skin | organizations | ‘np1’ ‘and safety’ ‘np2’ |
| Concept2: | file | protocol | ‘np1’ ‘stream’ ‘np2’ |
| ‘File Transfer’ | hypertext | stack | ‘np1’ ‘transfer’ ‘np2’ |
| | FTP | technology | ‘np2’ ‘cable’ ‘np1’ |
| Concept3: | internet | providers | ‘np1’ ‘service’ ‘np2’ |
| ‘Internet Service’ | phone | web sites | ‘np1’ ‘access’ ‘np2’ |
| | application | roots | ‘np1’ ‘hosting’ ‘np2’ |
| Concept4: | discount | store | ‘np1’ ‘food’ ‘np2’ |
| ‘Shopping’ | shop | service | ‘np1’ ‘and nutrition’ ‘np2’ |
| | grocery | products | ‘np1’ ‘supplement’ ‘np2’ |

in computer network connection logs and specifically [23] spots anomalies in time-evolving social networks as well. Last but not least, the PARAFAC decomposition has been used in community detection, where we have different views of the same network of people [24], or the network evolves

Table 18 Discovered factors from HATEN2-Tucker on the NELL-2 dataset

| | NP S1: Health | NP S2: Credit | NP S3: Network | NP S4: Algorithm | NP S5: Project | NP S6: Information |
|--------------|---------------------------|--------------------------|------------------------|-----------------------------|-------------------------------|------------------------------|
| Noun Phrase1 | health | credit | internet | optimization | agency | information |
| | child | charge | phone | rankings | proposal | details |
| | skin | bank | email | listings | management | news |
| | eye | ID | contact | algorithms | activities | material |
| | patient | account | network | indexing | manager | pictures |
| | NP O1: Region | NP O2: Service | NP O3: Web search | NP O4: Research | NP O5: Loan | NP O6: Network |
| Noun Phrase2 | world | providers | search | research | loan | roots |
| | state | system | website | experience | rates | speeds |
| | planet | service | page | work | mortgage | proxies |
| | region | insurance | industry | training | lender | ports |
| | globe | organization | performance | study | refinancing | routers |
| | Context C1: care | Context C2: credit | Context C3: function | Context C4: transfer | Context C5: support | Context C6: service |
| Context | 'np1' 'care' 'np2' | 'np1' 'card' 'np2' | 'np2' 'engine' 'np1' | 'np1' 'stream' 'np2' | 'np2' 'project' 'np1' | 'np1' 'service' 'np2' |
| | 'np1' 'insurance' 'np2' | 'np1' 'report' 'np2' | 'np2' 'returned' 'np1' | 'np1' 'transfer' 'np2' | 'np2' 'and development' 'np1' | 'np1' 'access np2' |
| | 'np1' 'service' 'np2' | 'np2' 'management' 'np1' | 'np2' 'results' 'np1' | 'np1' 'communication' 'np2' | 'np2' 'funding' 'np1' | 'np1' 'hosting' 'np2' |
| | 'np1' 'safety' 'np2' | 'np1' 'account' 'np2' | 'np2' 'returns' 'np1' | 'np1' 'protocol' 'np2' | 'np1' 'sponsoring' 'np2' | 'np1' 'broadband' 'np2' |
| | 'np1' 'and fitness' 'np2' | 'np1' 'debt' 'np2' | 'np2' 'machine' 'np1' | 'np2' 'cable' 'np1' | 'np1' 'supporting' 'np2' | 'np1' 'infrastructure' 'np2' |

Table 19 Concept discovery result using HATEN2-Tucker on the NELL-2 dataset

| Concepts | Noun Phrase1 | Noun Phrase2 | Context |
|--|--------------|---------------|--------------------------|
| Concept1: (S1, O2, C1) 'Health Care System' | health | providers | 'np1' 'care' 'np2' |
| | child | system | 'np1' 'insurance' 'np2' |
| | skin | professionals | 'np1' 'service' 'np2' |
| Concept2: (S3, O2, C6) 'Internet Service' | internet | providers | 'np1' 'service' 'np2' |
| | application | system | 'np1' 'access' 'np2' |
| | email | professionals | 'np1' 'hosting' 'np2' |
| Concept3: (S6, O2, C6) 'Information Access' | information | providers | 'np2' 'service' 'np1' |
| | details | system | 'np2' 'access' 'np1' |
| | news | professionals | 'np2' 'hosting' 'np1' |
| Concept4: (S4, O3, C3) 'Web Search Algorithm' | optimization | search | 'np2' 'engine' 'np1' |
| | rankings | website | 'np2' 'returned' 'np1' |
| | marketing | performance | 'np2' 'results' 'np1' |
| Concept5: (S5, O4, C5) 'Research Project Funding' | agency | research | 'np2' 'projects' 'np1' |
| | grants | training | 'np2' 'funding' 'np1' |
| | proposal | study | 'np1' 'sponsoring' 'np2' |

over time, and we are interested in identifying communities over time [25].

6.2 Tucker

In [26], apart from a highly memory-efficient Tucker decomposition algorithm, there is an overview of the various aspects of the Tucker decomposition as a data mining tool. The authors of [27] use a tensor in order to represent multiple semantic relations (such as ‘synonym’ or ‘antonym’) and use Tucker as a higher-order generalization of SVD, in order to perform latent semantic analysis. One of the most widely used Tucker variation is the so-called higher-order singular value decomposition (HOSVD) [28] which is a Tucker3 model with additional orthonormality constraints on the factor matrices. An exemplary work of employing HOSVD is [29] where the authors provide web search recommendations to users. HOSVD has been extensively used in Computer Vision applications [30,31].

6.3 Scalable algorithms for tensor analysis

Bader and Kolda develop efficient algorithms for sparse tensors [32], where they avoid the materialization of very large, unnecessary intermediate Khatri–Rao products. Kang et al. [18] proposed GigaTensor that first uses a distributed system for PARAFAC decomposition. GigaTensor is similar to HATEN2-PARAFAC-DRN in this paper; however, in this work we provide a significant improvement upon [18]. It can be shown that the ways that GigaTensor [18] and [32] avoid the intermediate data explosion are equivalent; however, GigaTensor [18] provides an algorithm which is optimized for the distributed setting. In the preliminary version of this paper, Jeon et al. [10] unify the large-scale Tucker and PARAFAC tensor decomposition algorithms on MAPREDUCE into a general framework, but do not consider the nonnegativity constraint. Beutel et al. [33] propose FlexiFaCT, a MAPREDUCE algorithm based on distributed stochastic gradient descent for PARAFAC and coupled PARAFAC decompositions. Bro and Sidiropoulos [34] use Tucker to compress a tensor, then do the PARAFAC decomposition on the compressed tensor, and finally, decompress the factors, thus speeding up the PARAFAC decomposition. An alternative approach, DBN, is introduced in [35] where the authors use relational algebra to break down the tensor into smaller tensors, using relational decomposition and thus achieving scalability. Furthermore, [23] introduces ParCube, an approximate and highly parallelizable algorithm for sparse PARAFAC decomposition. For scalable Tucker decomposition, there exists several previous works. Kolda and Sun [26] propose memory-efficient Tucker (MET) for scalable Tucker decomposition algorithm running on MAT-

LAB. Finally, Erdos and Miettinen introduce a scalable Boolean tensor decomposition using random walks [36].

7 Conclusion

In this paper, we propose HATEN2, a distributed method for large-scale tensor decompositions that runs on the MAPREDUCE platform. HATEN2 provides a unified framework to devise efficient MAPREDUCE algorithms for unconstrained and nonnegativity-constrained Tucker and PARAFAC tensor decompositions, which significantly reduces the intermediate data size and the running time. By careful design and implementation, HATEN2 decomposes up to 1000× larger tensors compared to existing methods. Furthermore, HATEN2 scales up near linearly on the number of machines. By applying HATEN2, we discover interesting patterns on various real-world data—knowledge bases, network traffic logs, and phone call history—with millions of rows, columns, and entries which were hard to analyze by existing methods.

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Compliance with ethical standards

Conflict of interest There are no potential conflicts of interests.

Human and animals rights statement The research does not involve human participants and/or animals.

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